1/2/30. The grid to the right consists of 74 unit squares, marked by gridlines. Partition the grid into five regions along the gridlines so that the areas of the regions are 1, 13, 19, 20, and 21. A square with a number should be contained in the region with that area.

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

				19					
		19					13		
	21	19	1	13	20	19	20	19	
		21					13		
					21				

Solution

			19	19	19	19			
	21	21	19	19	13	19	19	19	
21	21	19	19	13	13	13	13	19	20
21	21	19	13	13	20	19	19	19	20
21	21	19	1	13	20	19	20	19	20
21	21	21	21	13	20	20	20	20	20
21	21	21	21	13	13	13	13	20	20
	21	21	21	21	21	20	20	20	
			20	20	20	20			

2/2/30. Given a set of positive integers R, we define the *friend set* of R to be all positive integers which are divisible by at least one number in R. The friend set of R is denoted by $\mathcal{F}(R)$. A set G is called *unfriendly* if no element of G is a divisor of another element of G.

Let S_1 and S_2 be unfriendly sets. Suppose that $\mathcal{F}(S_1) = \mathcal{F}(S_2)$. Show that $S_1 = S_2$.

Solution

Suppose for the sake of contradiction that $S_1 \neq S_2$. Let n be the smallest positive integer that lies in exactly one of S_1 and S_2, and without loss of generality, assume that n lies in S_1 but not S_2. Then n lies in $\mathcal{F}(S_1)$, so n must also lie in $\mathcal{F}(S_2)$. This means that one of the elements of S_2 must be a divisor of n. Since n does not lie in S_2, some proper divisor d of n must lie in S_2. However, since $d < n$ and n is the smallest positive integer that lies in exactly one of S_1 and S_2, we see that d must also lie in S_1. This contradicts the condition that S_1 is an unfriendly set because S_1 contains both n and d, which is a divisor of n. So, $S_1 = S_2$.

3/2/30. Alice, Bob, and Chebyshev play a game. Alice puts six red chips into a bag, Bob puts seven blue chips into the bag, and Chebyshev puts eight green chips into the bag. Then, the almighty Zan removes chips from the bag one at a time and gives them back to the corresponding player. The winner of the game is the first player to get all of their chips back. Find, with proof, the probability that Bob wins the game.

Solution

Bob wins the game if and only if Zan removes the last blue chip before he removes the last red chip and before he removes the last green chip. We compute this probability using inclusion-exclusion.

We first compute the probability that Zan removes the last of the 7 blue chips before the last of the 6 red chips. This occurs if and only if, of the 13 red and blue chips, the last of them to be removed is red. Each of the 13 chips is equally likely to be the last of them to be removed, so the desired probability is $\frac{6}{13}$.

Similarly, the probability that Zan removes the last of the 7 blue chips before the last of the 8 green chips is $\frac{8}{7+8} = \frac{8}{15}$. Treating the 6 red chips and the 8 green chips as 14 indistinguishable chips, we see that the probability that Zan removes the last blue chip before *either* the last red chip or the last green chip is $\frac{14}{14+7} = \frac{2}{3}$.

Therefore, by inclusion-exclusion, the probability that Zan removes the last blue chip before *both* the last red chip and the last green chip is

$$\frac{6}{13} + \frac{8}{15} - \frac{2}{3} = \boxed{\frac{64}{195}},$$

which is approximately 32.8%.

4/2/30. Find, with proof, all ordered pairs of positive integers (a, b) with the following property: there exist positive integers r, s, and t such that for all n for which both sides are defined,

$$\binom{\binom{n}{a}}{b} = r\binom{n+s}{t}.$$

Solution

We treat both sides of the given equation as polynomials in n, where for any real number n, we define $\binom{n}{k}$ by the formula

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

Because these polynomials are equal for infinitely many values of n, they must be identical.

We first rule out the cases $a = 1$ and $b = 1$. If $a = 1$, then the given equation reduces to

$$\binom{n}{b} = r\binom{n+s}{t}.$$

As polynomials in n, the left-hand side has degree b while the right-hand side has degree t, so we must have $b = t$, and so

$$\binom{n}{b} = r\binom{n+s}{b}$$

for all n. But for any positive integer $n > b$, we have $\binom{n}{b} < \binom{n+s}{b} \le r\binom{n+s}{b}$, a contradiction. If $b = 1$, then the given equation reduces to $\binom{n}{a} = r\binom{n+s}{t}$, and similar reasoning applies. Thus, we must have $a \ge 2$ and $b \ge 2$.

Now, expanding both sides of the given equation, we get

$$\binom{\binom{n}{a}}{b} = \frac{1}{b!}\binom{n}{a}\left[\binom{n}{a} - 1\right]\left[\binom{n}{a} - 2\right]\cdots\left[\binom{n}{a} - (b-1)\right]$$

and

$$r\binom{n+s}{t} = \frac{r}{t!}(n+s)(n+(s-1))\dots(n+(s-t+1)).$$

Note that all of the roots of the polynomial $r\binom{n+s}{t}$ are distinct integers. Therefore, all the roots of $\binom{\binom{n}{a}}{b}$ must be distinct integers as well, which means that for each integer $0 \le i \le b-1$, the equation $\binom{n}{a} - i = 0$ must have a distinct integer roots.

Note that, for all n,

$$\begin{aligned}\binom{a-1-n}{a} &= \frac{(a-1-n)(a-2-n)\cdots(1-n)(-n)}{a!} \\ &= (-1)^a\frac{(n-a+1)(n-a+2)\cdots(n-1)(n)}{a!} \\ &= (-1)^a\binom{n}{a}.\end{aligned}$$

Because $\binom{0}{a} = \binom{1}{a} = \cdots = \binom{a-1}{a} = 0$, $\binom{a}{a} = 1$, and $3 = a + 1 \leq \binom{a+1}{a} < \binom{a+2}{a} < \ldots$, it follows that the only values of n for which $\binom{n}{a}$ could possibly equal 1 are $n = a$ and $n = a - 1 - a = -1$ (considering $\binom{n}{a}$ as a polynomial in n). Thus, $\binom{n}{a} - 1 = 0$ has at most two integer roots, so $a \leq 2$. Furthermore, we see that $\binom{n}{a} - 2 = 0$ has no integer roots at all, so $b \leq 2$.

Since we proved that $a \geq 2$ and $b \geq 2$, the only possible pair (a, b) is $(2, 2)$. We have

$$\begin{aligned}
\binom{\binom{n}{2}}{2} &= \frac{1}{2}\left(\frac{n(n-1)}{2}\right)\left(\frac{n(n-1)}{2} - 1\right) \\
&= \frac{1}{8}n(n-1)(n^2 - n - 2) \\
&= \frac{1}{8}n(n-1)(n+1)(n-2) \\
&= 3 \cdot \frac{(n+1)n(n-1)(n-2)}{4!} \\
&= 3\binom{n+1}{4}
\end{aligned}$$

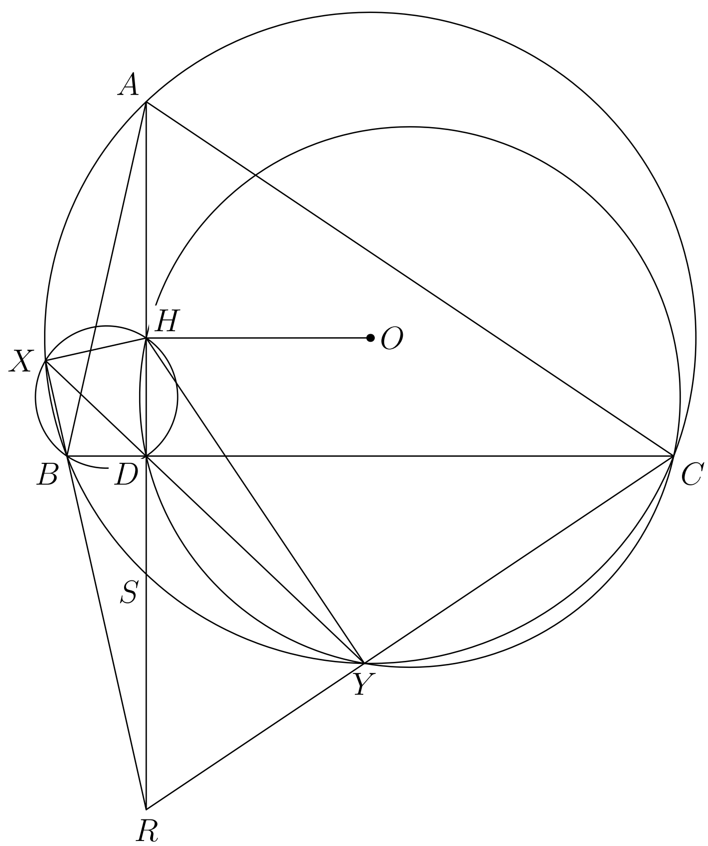
for all n, so the pair $(a, b) = (2, 2)$ satisfies the condition, and is the only such pair.

5/2/30. Acute scalene triangle $\triangle ABC$ has circumcenter O and orthocenter H. Points X and Y, distinct from B and C, lie on the circumcircle of $\triangle ABC$ such that $\angle BXH = \angle CYH = 90°$. Show that if lines XY, AH, and BC are concurrent, then $\overline{OH}$ is parallel to $\overline{BC}$.

Solution

Let D be the foot of the altitude from A to $\overline{BC}$, so D lies on line XY. Since $\angle BDH = \angle BXH = 90°$, D and X lie on the circle with diameter $\overline{BH}$. Similarly, D and Y lie on the circle with diameter $\overline{CH}$.

Let lines BX and CY intersect at R. Note that line BX is the radical axis of the circumcircles of triangles ABC and BXH, and line CY is the radical axis of the circumcircles of triangles ABC and CYD, so R is the radical center of all three circumcircles. Therefore, R lies on the radical axis of the circumcircles of triangles BXD and CYD, which is line AHD.



Note that D is the foot of the perpendicular from H to line BC, X is the foot of the perpendicular from H to line BR, and Y is the foot of the perpendicular from H to line CR. Since D, X, and Y are collinear, H must lie on the circumcircle of triangle BCR, with line XDY as the Simson line of point H with respect to triangle BCR.

Let line $AHDR$ intersect the circumcircle of triangle ABC again at S. We cite the well-known fact that $DS = DH$, so H and S are reflections in line BC. Then $\angle BHC = \angle BSC = 180^\circ - \angle A$, so $\angle BRC = 180^\circ - \angle BHC = \angle A$. Along with the fact that R lies on line AD, this implies that triangles ABC and RBC are congruent.

By Power of a Point,

$$RS \cdot RA = RB \cdot RX = RD \cdot RH.$$

Since $RA = 2RD$, $RH = 2RS$. But $RH = RS + SH$, so $RS = SH$. Also, $RS = AH$, so $AH = SH$. In other words, H is the midpoint of chord $\overline{AS}$ in the circumcircle of triangle ABC, so $\overline{OH}$ is perpendicular to $\overline{AS}$. Therefore, $\overline{OH}$ is parallel to $\overline{BC}$.

Problems by Shyan Ahkmal, David Altizio, Michael Tang, Sam Vandervelde, and USAMTS Staff.