USA Mathematical Talent Search

Round 2 Solutions

Year 29 — Academic Year 2017-2018

`www.usamts.org`

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.
2. Put your name and USAMTS ID# on **every page you submit**.
3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.
4. Submit your solutions by November 27, 2017, via one (and only one!) of the methods below:
   (a) Web: Log on to `www.usamts.org` to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 8 PM Eastern / 5 PM Pacific on November 27, 2017
   (b) Mail: USAMTS
   P.O. Box 4499
   New York, NY 10163
   (Solutions must be postmarked on or before November 27, 2017.)
5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto `www.usamts.org` and visiting the "My USAMTS" pages.
7. Round 2 results will be posted at `www.usamts.org` when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on `www.usamts.org`.

Each problem is worth 5 points.

1/2/29. Given a rectangular grid with some cells containing one letter, we say a row or column is *edible* if it has more than one cell with a letter and all such cells contain the same letter. Given such a grid, the hungry, hungry letter monster repeats the following procedure: he finds all edible rows and all edible columns and simultaneously eats all the letters in those rows and columns, removing those letters from the grid and leaving those cells empty. He continues this until no more edible rows and columns remain. Call a grid a *meal* if the letter monster can eat all of its letters using this procedure.

U		T			S	S
T		T		T	T	
A		A			S	A
A	T	A		M	S	
			M			
		T	U	M		
		A		M		

In the 7 by 7 grid to the right, fill each empty space with one letter so that the grid is a meal and there are a total eight Us, nine Ss, ten As, eleven Ms, and eleven Ts. Some letters have been given to you.

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

U	T	T	U	M	S	S
T	T	T	U	T	T	T
A	A	A	U	A	S	A
A	T	A	U	M	S	S
M	M	M	M	M	M	M
U	T	T	U	M	S	S
A	A	A	U	M	S	S

2/2/29. Let b be a positive integer. Grogg writes down a sequence whose first term is 1. Each term after that is the total number of digits in all the previous terms of the sequence when written in base b. For example, if $b = 3$, the sequence starts $1, 1, 2, 3, 5, 7, 9, 12, \ldots$. If $b = 2521$, what is the first positive power of b that does not appear in the sequence?

Solution

First, we look for a pattern in how the sequence increases. It increases by 1 with each term until we hit b. Then, the sequence increases by 2 with each term until we hit b^2. In general, once the sequence has passed b^{k-1}, it will increase by k each term until it passes b^k. This means that, if b^{r-1} is in the sequence, b^r will be in the sequence if and only if $b^r - b^{r-1}$ is a multiple of r. Or, equivalently, iff $b^r \equiv b^{r-1} \pmod{r}$. Since b^0 is in the sequence by definition, we're looking for the first power of b such that $b^r \not\equiv b^{r-1} \pmod{r}$. Since $b = \text{lcm}(1, 2, \ldots, 10) + 1$, we know that $b \equiv 1 \pmod{k}$ for $2 \le k \le 10$. So, the first time this equivalence might fail is at $r = 11$. And indeed it does fail at $r = 11$, since $2521 \equiv 2 \pmod{11}$ and

$$2^{11} - 2^{10} = 2^{10} \equiv 1 \pmod{11}.$$

Thus, $\boxed{b^{11}}$ is the first power of b that does not appear in the sequence.

Note: It's easiest to gain an understanding of what's going on by trying $b = 10$. Trying simpler problems that you understand better is an important problem solving strategy. In this case, this problem actually came from first trying the case for $b = 10$.

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3/2/29. The USAMTS tug-of-war team needs to pick a representative to send to the national tug-of-war convention. They don't care who they send, as long as they don't send the weakest person on the team. Their team consists of 20 people, who each pull with a different constant strength. They want to design a tournament, with each round planned ahead of time, which at the end will allow them to pick a valid representative. Each round of the tournament is a 10-on-10 tug-of-war match. A round may end in one side winning, or in a tie if the strengths of each side are matched. Show that they can choose a representative using a tournament with 10 rounds.

Solution

Consider the following cyclic sequence of games. In each game, the top 10 players play against the bottom 10 in the list.

Player 1	Player 2		Player 10	Player 11
Player 2	Player 3		Player 11	Player 12
Player 3	Player 4		Player 12	Player 13
Player 4	Player 5		Player 13	Player 14
Player 5	Player 6		Player 14	Player 15
Player 6	Player 7		Player 15	Player 16
Player 7	Player 8		Player 16	Player 17
Player 8	Player 9		Player 17	Player 18
Player 9	Player 10		Player 18	Player 19
Player 10	Player 11		Player 19	Player 20
Player 11	Player 12	...	Player 20	Player 1
Player 12	Player 13		Player 1	Player 2
Player 13	Player 14		Player 2	Player 3
Player 14	Player 15		Player 3	Player 4
Player 15	Player 16		Player 4	Player 5
Player 16	Player 17		Player 5	Player 6
Player 17	Player 18		Player 6	Player 7
Player 18	Player 19		Player 7	Player 8
Player 19	Player 20		Player 8	Player 9
Player 20	Player 1		Player 9	Player 10

We get each successive list by moving the player on the top of the previous list to the bottom of the next list. Note that there are a total of 11 games here, but the last game is exactly the same as the first, so we know the results of all 11 games by only playing the first 10. Note that if the first game is a tie, the second game determines which of player 1 and player 11 is stronger. So, without loss of generality, we'll assume the bottom team lost the first game.

In each successive game, one player on each team switches spots. So, the first time the bottom team wins a game, we know that the player that switched to the bottom is *stronger* than the person she swapped with.

So, we will have a representative if the bottom team ever wins a game. But, we know the bottom team wins the 11th game because it's the exact *opposite* of the first game! So, we must be able to find a representative in at most 10 games.

Note: While we expect it is the case, the authors do not know if 10 games is optimal.

4/2/29. Zan starts with a rational number $0 < \frac{a}{b} < 1$ written on the board in lowest terms. Then, every second, Zan adds 1 to both the numerator and denominator of the latest fraction and writes the result in lowest terms. Zan stops as soon as he writes a fraction of the form $\frac{n}{n+1}$, for some positive integer n. If $\frac{a}{b}$ started in that form, Zan does nothing.

As an example, if Zan starts with $\frac{13}{19}$, then after one second he writes $\frac{14}{20} = \frac{7}{10}$, then after two seconds $\frac{8}{11}$, then $\frac{9}{12} = \frac{3}{4}$, at which point he stops.

(a) Prove that Zan will stop in less than $b - a$ seconds.

(b) Show that if $\frac{n}{n+1}$ is the final number, then

$$\frac{n-1}{n} < \frac{a}{b} \leq \frac{n}{n+1}.$$

Solution

(a) We go by induction on $b-a$. The base case $b-a = 1$ is done by definition, because Zan won't make any moves. Suppose the fraction is $\frac{a}{b}$ and the claim is true for all values less than $b-a$. Let m be the smallest positive integer with $\gcd(a+m, b+m) > 1$, and define $d = \gcd(a+m, b+m)$. Note that we have $m < d$, since otherwise we could write $m = d + k$ and $m' = m - d$ would be a smaller solution. The definition of m means after m steps, we get a fraction whose numerator and denominator differ by $\frac{b-a}{d}$, which is an integer since

$$d = \gcd(a+m, b+m) = \gcd(a+m, [b+m] - [a+m]) = \gcd(a+m, b-a).$$

By the induction hypothesis, it takes at most $\frac{b-a}{d} - 1$ steps to finish after these m steps. So to complete the induction we need to show that

$$m + \frac{b-a}{d} - 1 < b - a.$$

Since d and $\frac{b-a}{d}$ are positive integers, we can say that

$$(d-1)\left(\frac{b-a}{d} - 1\right) \geq 0.$$

Expanding and rearranging, this becomes

$$d + \frac{b-a}{d} - 1 \leq b - a.$$

Since $m < d$, this implies the inequality we wanted to show.

(b) We will assume that Zan's sequence has more than one term, since otherwise the result is obvious. First, we claim that Zan's sequence is strictly increasing. That is, $\frac{a+1}{b+1} > \frac{a}{b}$. Clearing denominators, this is equivalent to

$$b(a+1) - a(b+1) = b - a > 0,$$

which is true since $\frac{a}{b} < 1$. This means that $\frac{a}{b} < \frac{n}{n+1}$, so it now suffices to show that $\frac{n-1}{n} < \frac{a}{b}$. To that end, suppose toward a contradiction that $\frac{a}{b} < \frac{k}{k+1}$, where $k < n$. Clearing the denominators in our condition, we get

$$ak + a < kb.$$

Adding k to both sides and using that $x < y \Rightarrow x+1 \leq y$ for integers x, y, this becomes

$$ak + a + k + 1 \leq kb + k.$$

After factoring the left side and dividing both sides by $(k+1)(b+1)$, we have

$$\frac{a+1}{b+1} \leq \frac{k}{k+1}.$$

This means that Zan's sequence is bounded by $\frac{k}{k+1}$, which is impossible if Zan's final number is $\frac{n}{n+1}$. So, we must have

$$\frac{n-1}{n} < \frac{a}{b} \leq \frac{n}{n+1}.$$

5/2/29. There are n distinct points in the plane, no three of which are collinear. Suppose that A and B are two of these points. We say that segment AB is *independent* if there is a straight line such that points A and B are on one side of the line, and the other $n-2$ points are on the other side. What is the maximum possible number of independent segments?

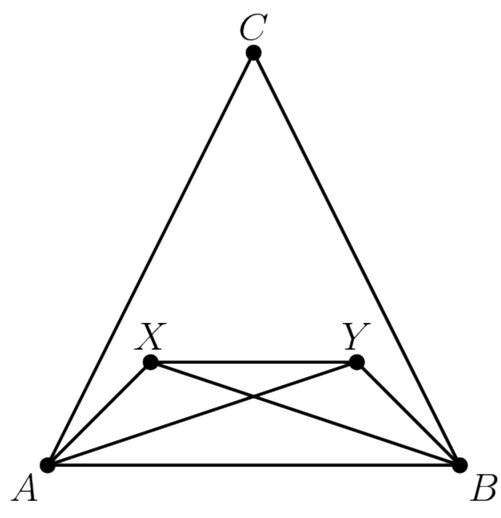
Solution (REVISED based on input by Kevin Ren.)

If $n = 1$, there are $\boxed{0}$ independent segments. If $n = 2$, there is exactly $\boxed{1}$ independent segment. If $n = 3$, there are exactly $\boxed{3}$ independent segments (the edges of the triangle formed by the three points).

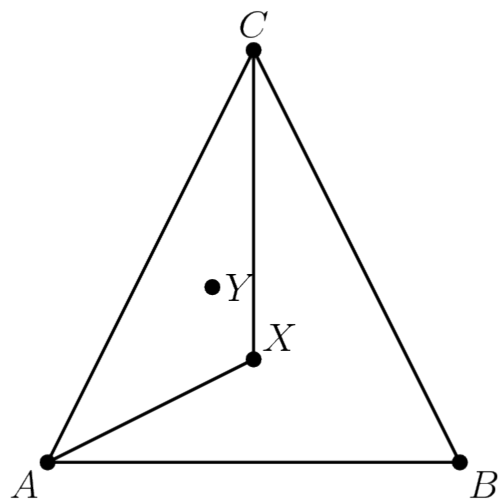
If $n \geq 4$, let $H = H_1H_2 \ldots H_k$ be the convex hull of the given points and $G_1, G_2, \ldots, G_l$ be all given points that are not vertices of H. We will need two cases, first when H is a triangle, and second when H is not a triangle.

First assume the convex hull of the set of points is a triangle ABC. No independent segment can have both of its endpoints in the interior of H. We claim that for any edge, say AB, of this triangle there is at most one point X in the interior of ABC such that both XA and XB are independent.

To that end, suppose that we have two points X and Y in the interior of ABC such that AX, BX and AY, BY are independent. First, we note that if AY and BX cross, then neither of them can be independent. To see this note that in this case $AXYB$ is a convex quadrilateral. The segment that separated AY from the other points would have to pass through AX, XY, and AB. The only such line is the diagonal BX, which is impossible because A and Y are not on the same side of that diagonal. Similarly, BX cannot be independent.



So, AY and BX do not cross. Without loss of generality, Y is contained in the interior of AXC. In this case, a segment that separates BY from A, C, and X must pass through AY, CY, and XY. But that is impossible, because any such line passes through Y. This proves the claim.

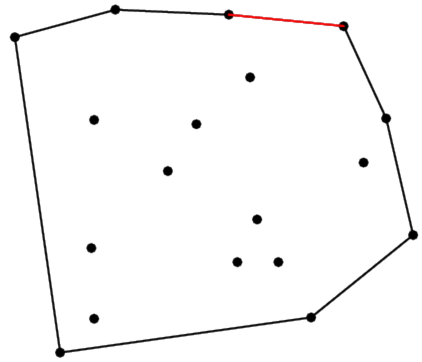


Therefore, we can bound the number of independent segments when the convex hull is a triangle. First, if there is an interior point X such that AX, BX, and CX are all independent, then the other $n-4$ interior points are part of at most one independent segment, and each side of the convex hull is an independent segment. This gives at most $3+3+(n-4)=n+2$ independent segments.

Alternatively, if there is no interior point X such that AX, BX, and CX are independent, then there are at most three interior points incident to two independent segments, one from each side. This gives 6 independent segments. The remaining $n-6$ interior points are part of at most one independent segment, and each side of the convex hull is an independent segment. So, we see that there are at most $3+3\cdot 2+(n-6)=n+3$ independent segments in this case. All together, we can say that there are at most $n+3$ independent segments when the convex hull is a triangle.

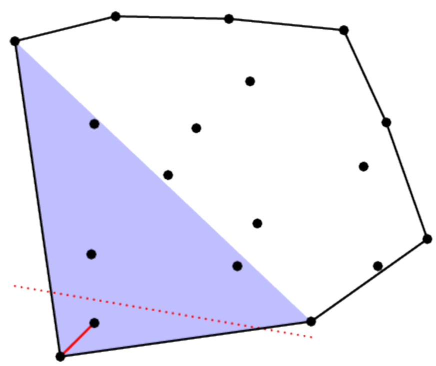
Next, we consider the case that the convex hull is not a triangle. Let $H = H_1H_2\dots H_k$ be the convex hull of the given points and G_1, G_2, $\dots$, G_l be all given points that are not vertices of H.

If both of the endpoints of an independent segment are vertices of H then it must be a side of H:



There are at most k such independent segments.

Next we look at independent segments with one endpoint an H_i and the other a G_j. If H_iG_j is independent, then G_j lies in the interior of $H_{i-1}H_iH_{i+1}$ (H-indices run modulo k).



Since no three such triangles have a common interior point, no G_j is an endpoint of more than two independent segments. If G_j is an endpoint of two independent segments, we say that it is a *double*. Let m be the number of doubles.

If G_j is a double, then it must lie in the interior of the intersection δ_i of $H_{i-1}H_iH_{i+1}$ and $H_iH_{i+1}H_{i+2}$ for some i.

Suppose that some δ_i contains two doubles P and Q. Since $H_{i-1}H_iH_{i+1}H_{i+2}$ is a convex quadrilateral, the line PQ does not intersect at least one of the segments $H_{i-1}H_i$, H_iH_{i+1}, and $H_{i+1}H_{i+2}$, say $H_{i-1}H_i$. This means that (with P and Q's labels switched if need be) $H_{i-1}PQH_i$ is a convex quadrilateral. Therefore, the segment $H_{i-1}Q$ intersects the segment H_iP in an interior point and H_iP cannot be an independent segment, a contradiction.

Therefore, the number of doubles is at most the number of δs. Since there are l Gs and k δs, this means that the number of independent segments involving Gs is bounded above by $l + \min\{k, l\}$. Adding in the edges of the convex hull, we see that the total number of independent segments is at most $k + l + \min\{k, l\}$ when the convex hull is not a triangle. Since $k + l = n$, this is at most $n + \lfloor \frac{n}{2} \rfloor$.

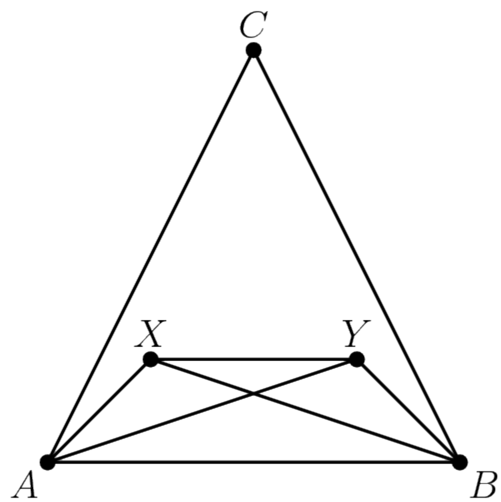
This gives us a bound of $\max(n + 3, n + \lfloor \frac{n}{2} \rfloor)$. For $n > 5$, this bound is simply $n + \lfloor \frac{n}{2} \rfloor$ and is attained when $k = \lceil \frac{n}{2} \rceil$ and each G_j lies in the interior of δ_j.

To complete the proof, we just have to wrestle with the cases $n = 4$ and $n = 5$.

If $n = 4$, there are at most $\boxed{6}$ independent segments, achieved by placing a single point inside a triangular convex hull.

Finally, we consider $n = 5$. If the convex hull is not a triangle, we know there are at most $n + \lfloor \frac{n}{2} \rfloor = 5 + 2 = 7$ independent segments.

We claim that no interior point can form an independent segment with all three vertices in a triangular convex hull ABC. To see this, note that there are exactly 2 vertices say X, Y in the interior of ABC. The line XY intersects exactly two of the sides of ABC, without loss of generality, assume that AB is the third side. Then, $AXYB$ forms a convex quadrilateral, as shown below.



We know from earlier that neither AY nor BX can be independent in this case, so we have the claim. Thus, there are at most $3 + 2 + 2 = \boxed{7}$ independent segments for $n = 5$. This is achieved in the picture above (AB, BC, CA, XA, XC, YB, and YC are independent).

Thus the maximum value is $\boxed{n + \left\lfloor \frac{n}{2} \right\rfloor}$ for $n \geq 4$, and $\boxed{0}$ for $n = 1$, $\boxed{1}$ for $n = 2$, and $\boxed{3}$ for $n = 3$.

(Problem proposed by Nikolai Beluhov)