USA Mathematical Talent Search

Round 3 Solutions

Year 28 — Academic Year 2016–2017

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1/3/28. Fill in each square of the grid with a number from 1 to 16, using each number exactly once. Numbers at the left or top give the largest sum of two numbers in that row or column. Numbers at the right or bottom give the largest difference of two numbers in that row or column.

+	13	28	25	23	
13					7
28					
					8
23					8
	7		12	12	−

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

+	13	28	25	23	
13	3	9	2	4	7
28	1	12	11	16	
	8	15	14	7	8
23	5	13	10	6	8
	7		12	12	−

2/3/28. Malmer Pebane, Fames Jung, and Weven Dare are perfect logicians that always tell the truth. Malmer decides to pose a puzzle to his friends: he tells them that the day of his birthday is at most the number of the month of his birthday. Then Malmer announces that he will whisper the day of his birthday to Fames and the month of his birthday to Weven, and he does exactly that.

After Malmer whispers to both of them, Fames thinks a bit, then says "Weven cannot know what Malmer's birthday is."

After that, Weven thinks a bit, then says "Fames also cannot know what Malmer's birthday is."

This exchange repeats, with Fames and Weven speaking alternately and each saying the other can't know Malmer's birthday. However, at one point, Weven instead announces "Fames and I can now know what Malmer's birthday is. Interestingly, that was the longest conversation like that we could have possibly had before both figuring out Malmer's birthday."

Find Malmer's birthday.

Solution

We imagine that Fames and Weven are each given a number between 1 and 12 (representing either the day or the month of Malmer's birthday respectively).

We analyze what happens at the first step, when Fames claims "Weven cannot know what Malmer's birthday is." Since Weven is given the month, and the day can be any number less than or equal to the month, he could only know Malmer's birthday if he were given a 1 (as this would imply Malmer's birthday was 1/1). So, Weven must not have a 1. In order for Fames to know that Weven doesn't have a 1, Fames also must not have a 1. So, we conclude that neither Fames nor Weven was given a 1.

Next, Weven claims that "Fames also cannot know what Malmer's birthday is." Since Fames is given the day, and the month can be any number greater than or equal to the day, he could only know Malmer's birthday if he were given a 12 (as this would imply Malmer's birthday was 12/12). So, Fames must not have a 12. In order for Weven to know that Fames doesn't have a 12, he also must not have a 12. So, we conclude that neither Fames nor Weven was given a 12.

Similarly, if this exchange were to happen again, Fames's statement would allow us to conclude that neither Fames nor Weven was given a 2, and Weven's statement would allow us to conclude that neither Fames nor Weven was given an 11.

In order for this to continue for as long as possible, both players must continue to eliminate numbers until Fames eliminates 6. Once this happens, both players know that only 7 is possible, and Malmer's birthday is $\boxed{(7, 7)}$ (July 7th).

3/3/28. An *n-city* is an $n \times n$ grid of positive integers such that every entry greater than 1 is the sum of an entry in the same row and an entry in the same column. Shown below is an example 3-city.

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 6 & 4 & 1 \end{pmatrix}$$

(a) Construct a 5-city that includes some entry that is at least 150. (It is acceptable simply to write the 5-city. You do not need to explain how you found it.)

(b) Show that for all $n \geq 1$, the largest entry in an n-city is at most $3^{\binom{n}{2}}$.

Solution

(a) An example of a 5-city with an entry that is at least 150 is shown below. You may have found a different example.

$$\begin{pmatrix} 1 & 2 & 5 & 3 & 4 \\ 26 & 1 & 6 & 25 & 10 \\ 67 & 69 & 1 & 15 & 12 \\ 326 & 259 & 138 & 1 & 11 \\ 52 & 121 & 127 & 26 & 1 \end{pmatrix}$$

(b) An n-city must have at least one 1 in every row and column. So an n-city must have at least n 1's, and hence can have at most $n^2 - n = 2\binom{n}{2}$ entries greater than 1. Let us list these entries as $a_1, a_2, \ldots, a_{2\binom{n}{2}}$ from least to greatest.

Notice that a_1 can be at most $1 + 1 = 2$. Then a_2 can be at most $a_1 + 1 = 3$, and a_3 can be at most $a_2 + a_1 = 3 + 2 = 5$. In general, we see that a_k is at most $a_{k-1} + a_{k-2}$. Hence, we conclude that $a_k \leq F_{k+2}$ for all k, where F_m is the m-th Fibonacci number.

Therefore, we have $a_{2\binom{n}{2}} \leq F_{2\binom{n}{2}+2}$. So, it suffices to show that $F_{2\binom{n}{2}+2} \leq 3^{\binom{n}{2}}$.

To that end, we claim that $F_{2m+2} \leq 3^m$ for all $m \geq 0$. For $m = 0$ both sides of the inequality are 1. Then,

$$\begin{aligned} F_{2m+2} &= F_{2m+1} + F_{2m} \\ &= 2F_{2m} + F_{2m-1} \\ &\leq 3F_{2m}. \end{aligned}$$

Hence, by induction $F_{2m+2} \leq 3(3^{m-1}) = 3^m$. Thus,

$$a_{2\binom{n}{2}} \leq F_{2\binom{n}{2}+2} \leq 3^{\binom{n}{2}}.$$

So, every entry in an n-city is at most $3^{\binom{n}{2}}$ as desired.

4/3/28. Let $A_1, \dots, A_n$ and $B_1, \dots, B_n$ be sets of points in the plane. Suppose that for all points x,

$$D(x, A_1) + D(x, A_2) + \cdots + D(x, A_n) \geq D(x, B_1) + D(x, B_2) + \cdots + D(x, B_n),$$

where $D(x, y)$ denotes the distance between x and y. Show that the A_i's and the B_i's share the same center of mass.

Solution

Lemma: If $a > b$, then $\sqrt{a^2 + b^2} - a \leq \frac{b^2}{2a}$.

Proof of Lemma: We see that

$$a^2 + b^2 \leq \frac{b^4}{4a^2} + a^2 + b^2,$$

and both sides are positive. Taking the square root of both sides, we get

$$\sqrt{a^2 + b^2} \leq \frac{b^2 + 2a^2}{2a}.$$

Subtracting a from both sides gives the desired claim. ■

Suppose that the centers of mass of the A_i and B_i are M_A and M_B respectively, and that d is a real number with $D(A_i, M_A) \leq d$ and $D(B_i, M_B) \leq d$ for all i. Let $D(M_A, M_B) = \ell$, and suppose for the sake of contradiction that $\ell > 0$. Choose a real number q such that

$$q > d + \frac{d^2}{2\ell}.$$

Let P be the point on line $M_A M_B$ with $D(P, M_A) = q$ and $D(P, M_B) = q + \ell$. We claim that

$$D(P, A_1) + D(P, A_2) + \cdots + D(P, A_n) < D(P, B_1) + D(P, B_2) + \cdots + D(P, B_n).$$

For each i, let A_i' be the projection of A_i onto line $M_A M_B$, and similarly define B_i'. We see that $PA_i'A_i$ is a right triangle with hypotenuse PA_i, so applying the Pythagorean Theorem gives us $D(P, A_i)^2 = D(P, A_i')^2 + D(A_i', A_i)^2$. Since A_i' is the closest point to A_i on the line $M_A M_B$, we have $D(A_i, A_i') \leq D(A_i, M_A) \leq d$. So, $D(P, A_i)^2 \leq D(P, A_i')^2 + d^2$. For brevity, let $u_i = D(P, A_i)$ and $v_i = D(P, A_i')$ for each i. We rewrite our inequality as $u_i^2 \leq d^2 + v_i^2$. Thus, $u_i \leq \sqrt{d^2 + v_i^2}$, and $u_i - v_i \leq \sqrt{d^2 + v_i^2} - v_i$. Applying the Lemma gives us

$$u_i - v_i \leq \frac{d^2}{2v_i}.$$

Applying the Triangle Inequality to triangle PA'_iM_A, we get
$D(P, A'_i) + D(A'_i, M_A) \geq D(P, M_A)$. Since $D(A'_i, M_A) \leq D(A_i, M_A) \leq d$, we have $v_i + d \geq q$. Combining this inequality with the inequality $q > d + \frac{d^2}{2\ell}$ and subtracting d, we get
$v_i \geq q - d > \frac{d^2}{2\ell}$.

Applying the inequality $v_i > \frac{d^2}{2\ell}$ to the right side of the inequality $u_i - v_i \leq \frac{d^2}{2v_i}$, we get $u_i - v_i < \ell$. So, $u_i < v_i + \ell$. And since $\sum_{i=1}^{n} v_i = nq$ by $D(P, M_A) = q$ and the definition of the center of mass, we have $\sum_{i=1}^{n} u_i < nq + n\ell$.

We also have $D(P, B_i) \geq D(P, B'_i)$, so

$$\sum_{i=1}^{n} D(P, B_i) \geq \sum_{i=1}^{n} D(P, B'_i) = nD(P, M_B) = nq + n\ell.$$

Therefore,

$$\sum_{i=1}^{n} D(P, A_i) < nq + n\ell \leq \sum_{i=1}^{n} D(P, B_i).$$

This is a contradiction of the condition, so $\ell = 0$ as desired.

5/3/28. Consider the set $S = \{q + \frac{1}{q},$ where q ranges over all positive rational numbers$\}$.

(a) Let N be a positive integer. Show that N is the sum of two elements of S if and only if N is the product of two elements of S.

(b) Show that there exist infinitely many positive integers N that cannot be written as the sum of two elements of S.

(c) Show that there exist infinitely many positive integers N that can be written as the sum of two elements of S.

Solution

(a) Note that the "right side implies left side" implication is true even without the requirement that N is an integer. Indeed, assume that $N = (q + \frac{1}{q}) \cdot (r + \frac{1}{r})$, with q, r rational. By expanding the product, we obtain:

$$N = qr + \frac{1}{qr} + \frac{q}{r} + \frac{r}{q} = \left(x + \frac{1}{x}\right) + \left(y + \frac{1}{y}\right), \text{ where } x = qr \text{ and } y = \frac{q}{r} \text{ are rational.}$$

We now show that the converse is true. Let N be an integer such that

$$N = (x + \frac{1}{x}) + (y + \frac{1}{y}),$$

where x, y are positive rational numbers. We can write x, y as irreducible fractions $x = \frac{a}{b}$ and $y = \frac{c}{d}$, with a, b, c, d positive integers satisfying $\gcd(a, b) = \gcd(c, d) = 1$. After expanding the formula for N we obtain:

$$N = \frac{(a^2 + b^2)cd + (c^2 + d^2)ab}{abcd}.$$

Since N is an integer, it follows that ab must divide the numerator. Thus, ab must divide $(a^2 + b^2)cd$. Since $\gcd(a, b) = 1$, it follows that $\gcd(ab, a^2 + b^2) = 1$. So, we must have that ab divides cd. Similarly, by using the fact that cd must also divide the numerator of the fraction, we obtain that cd divides ab. It follows $ab = cd$. Note now that if we let $p = \frac{c}{b}$ and $q = \frac{a}{c}$, then we have:

$$\begin{aligned}
\left(p + \frac{1}{p}\right)\left(q + \frac{1}{q}\right) &= \left(\frac{c}{b} + \frac{b}{c}\right)\left(\frac{a}{c} + \frac{c}{a}\right) \\
&= \frac{a}{b} + \frac{b}{a} + \frac{c^2}{ab} + \frac{ab}{c^2} \\
&= \frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c} \\
&= \left(x + \frac{1}{x}\right) + \left(y + \frac{1}{y}\right) \\
&= N.
\end{aligned}$$

We used above the equality $\frac{c^2}{ab} = \frac{c}{d}$, which is equivalent to $ab = cd$.

Remark. Another way to finish the proof from $ab = cd$ is to argue that there must exist positive integers s, t, u, v such that $a = st$, $b = uv$, $c = su$, and $d = tv$. It follows that $N = (\frac{s}{v} + \frac{v}{s})(\frac{t}{u} + \frac{u}{t})$.

(b) We show that if N is an integer divisible by 8, then it is not possible to write N as the sum of two elements of S. Equivalently, it suffices to show that N is not the product of two elements of S. (It is somewhat faster to work with the product than the sum.) Assume $N = (\frac{a}{b} + \frac{b}{a}) \cdot (\frac{c}{d} + \frac{d}{c})$, where a, b, c, d are positive integers such that $\frac{a}{b}$ and $\frac{c}{d}$ are irreducible, positive fractions. After expanding the product, we obtain

$$\frac{(a^2 + b^2) \cdot (c^2 + d^2)}{abcd} = N.$$

Since N is divisible by 8, it follows that 8 must divide the numerator $(a^2 + b^2) \cdot (c^2 + d^2)$. Thus, at least one of the two factors of the product must be divisible by 4. Assume for instance that 4 divides $a^2 + b^2$. By considering all the possibilities for $a, b \pmod 4$, it is easy to see that a, b must both be even. However, this contradicts the fact that the fraction $\frac{a}{b}$ is irreducible.

Remark. Another option is to show that if N has a prime factor p with $p \equiv 3 \pmod 4$, then N cannot be written as the product of two elements of S.

(c) We show that there exist infinitely many integers N of the form $N = (a + \frac{1}{a})(b + \frac{1}{b})$, with a, b positive integers. Note that this statement is stronger than the statement of (c). We have:

$$\left(a + \frac{1}{a}\right) \cdot \left(b + \frac{1}{b}\right) = \frac{a^2 + 1}{b} \cdot \frac{b^2 + 1}{a}.$$

We construct recursively infinitely many pairs of integers (a, b), with $a < b$ and such that a divides $b^2 + 1$, and b divides $a^2 + 1$. First, observe that the pair $(1, 2)$ works. Now, assuming that (a, b) satisfies the requirements, we show that $(b, \frac{b^2+1}{a})$ also works. Indeed, if we let $c = \frac{b^2+1}{a}$, then we know that c is an integer. The fact that $b < c$ is equivalent to $ab < b^2 + 1$, which is true since $a < b$. The fact that c divides $b^2 + 1$ follows from $b^2 + 1 = ac$. Finally, the fact that b divides $c^2 + 1$ is equivalent to b divides $\frac{(b^2+1)^2+a^2}{a^2}$. This is true because $(b^2 + 1)^2 + a^2 \equiv 1 + a^2 \equiv 0 \pmod b$, and $\gcd(a, b) = 1$.

Remark. The first few pairs obtained through our construction are $(1, 2), (2, 5), (5, 13), \ldots$. The corresponding values of N are $5, 13, 68, \ldots$.