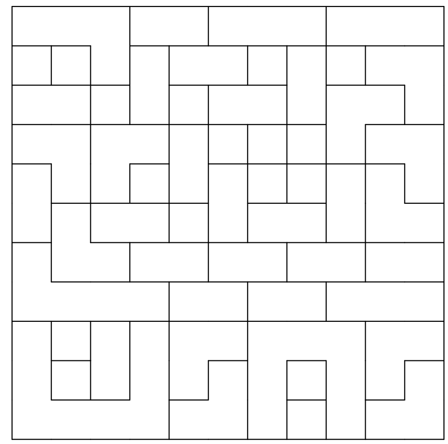
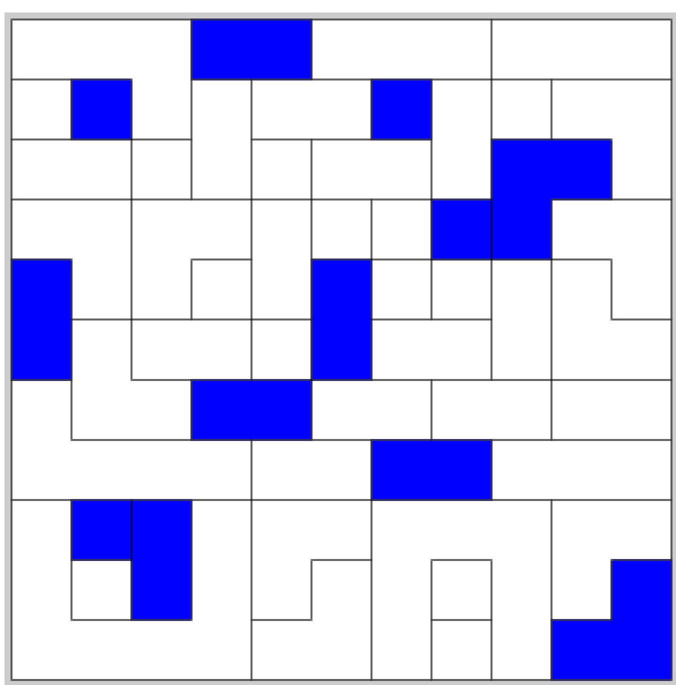
1/2/28. Shade in some of the regions in the grid to the right so that the shaded area is equal for each of the 11 rows and columns. Regions must be fully shaded or fully unshaded, at least one region must be shaded, and the area of shaded regions must be at most half of the whole grid.



You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution



2/2/28. Find all triples of three-digit positive integers $x < y < z$ with x, y, z in arithmetic progression and $x, y, z + 1000$ in geometric progression.

For this problem, you may use calculators or computers to gain an intuition about how to solve the problem. However, your final submission should include mathematical derivations or proofs and should not be a solution by exhaustive search.

Solution

Since x, y, and z are in arithmetic progression, we have

$$y = \frac{x+z}{2},$$

and since x, y, and $z + 1000$ are in geometric progression, we have

$$y = \sqrt{x(z + 1000)}.$$

Combining these equations gives us

$$\frac{x+z}{2} = \sqrt{x(z + 1000)}.$$

Rearranging and squaring gives us

$$x^2 + 2xz + z^2 = 4xz + 4000x,$$

or

$$(z - x)^2 = 4000x.$$

Since $4000 = 10 \cdot 20^2$, we know that $10x$ must be a square, and hence $x = 10n^2$ for some positive integer n. Therefore,

$$(z - 10n^2)^2 = (200n)^2,$$

or $z = 10n^2 + 200n = n(10n + 200)$.

Since z is a three-digit number, n must be at most 4. But $n \geq 4$ since x is also a three-digit number, so we must have $n = 4$. It follows that the only solution is
$(x, y, z) = \boxed{(160, 560, 960)}$.

3/2/28. Suppose m and n are relatively prime positive integers. A regular m-gon and a regular n-gon are inscribed in a circle. Let d be the minimum distance **in degrees** (of the arc along the circle) between a vertex of the m-gon and a vertex of the n-gon. What is the maximum possible value of d?

Solution

The answer is $\boxed{\dfrac{180^\circ}{mn}}$.

It is equivalent to consider each point on the circle as a number from 0 to 1, with each number treated mod 1. The regular m-gon becomes an infinite arithmetic progression with common difference $1/m$, and the regular n-gon becomes an infinite arithmetic progression with common difference $1/n$. Then our claimed answer is equivalent to $\frac{1}{2mn}$ being the largest value of the smallest difference of two terms of these progressions. The construction is simple: let the m-progression start at 0 and the n-progression start at $\frac{1}{2mn}$. Then every term in the m-progression is an integer multiple of $\dfrac{1}{mn}$ while each term in the n-progression is a half-integer times $\dfrac{1}{mn}$. Therefore, the difference between terms in different progressions is at least $\dfrac{1}{2} \cdot \dfrac{1}{mn}$.

To show this is best, define d to be the smallest distance between terms of the progression. We will show that $\left|d - \frac{1}{mn}\right|$ is a difference between their terms as well. Suppose $x - y = d$, WLOG with x in the m-progression and y in the n-progression (since if $x - y = -d$ we can use the terms $-x$ and $-y$ instead). Since m and n are relatively prime, we can use Bezout's lemma to see that there are integers p, q with $pm - qn = 1$. Consider the terms $x + q/m$ and $y + p/n$. We have

$$\left(x + \frac{q}{m}\right) - \left(y + \frac{p}{n}\right) = x - y - \frac{pm - qn}{mn} = d - \frac{1}{mn}.$$

So the difference between the terms is $\left|d - \frac{1}{mn}\right|$. Since d was chosen to be the smallest possible distance between terms of the progression, it must be the case that

$$d \leq \left|d - \frac{1}{mn}\right|,$$

which implies that $d \leq \frac{1}{2mn}$ as desired.

4/2/28. On Binary Island, residents communicate using special paper. Each piece of paper is a $1 \times n$ row of initially uncolored squares. To send a message, each square on the paper must either be colored either red or green. Unfortunately the paper on the island has become damaged, and each sheet of paper has 10 random consecutive squares each of which is randomly colored red or green.

Malmer and Weven would like to develop a scheme that allows them to send messages of length 2016 between one another. They would like to be able to send any message of length 2016, and they want their scheme to work with perfect accuracy. What is the smallest value of n for which they can develop such a strategy?

Note that when sending a message, one can see which 10 squares are colored and what colors they are. One also knows on which square the message begins, and on which square the message ends.

Solution

The answer is $2016 + 10 = \boxed{2026}$.

It's clear that no smaller n can work. Once 10 out of n squares have been filled in, there are only 2^{n-10} ways to fill in the remaining squares. Since Weven and Malmer want to be able to send 2^{2016} different messages, we must have $n - 10 \geq 2016$.

Now we present a strategy that allows Weven and Malmer to send a message of length 2016 when $n = 2026$.

Number the squares from 1 to 2026. Divide the squares into 10 sets of squares, where set i consists of the squares whose number is congruent to i mod 10. Observe that each set contains exactly one initially colored square. So it suffices to find a strategy to transmit a message of length $k - 1$ given k squares with 1 square initially covered. Malmer and Weven can then use this strategy to send 10 interleaved submessages with a total length of $2026 - 10 = 2016$.

Malmer and Weven will use the following decoding scheme:

(i) The message is least k squares.

(ii) If the first square is green, flip all the other squares and read that message.

(iii) If the first square is red, leave all the other squares as is and read that message.

Then we can encode our message as follows:

(i) If the bad square is first:

(a) If the first square is red: then fill the remaining squares with what you want your message to be.

(b) If the first square is green: then flip the colors of your message and fill the remaining squares.

(ii) If the bad square is in the middle:

(a) If the bad square is in the color you desire it to be already: color the first square red and fill the remaining squares with what you want your message to be.

(b) If the bad square is not the color you desire it to be already: color the first square green, flip all the colors of your message and fill the remaining squares.

5/2/28. Let $n \geq 4$ and $y_1, \ldots, y_n$ real with

$$\sum_{k=1}^{n} y_k = \sum_{k=1}^{n} k y_k = \sum_{k=1}^{n} k^2 y_k = 0$$

and

$$y_{k+3} - 3y_{k+2} + 3y_{k+1} - y_k = 0$$

for $1 \leq k \leq n-3$. Prove that

$$\sum_{k=1}^{n} k^3 y_k = 0.$$

Solution

The second condition implies that there exists a quadratic $f(x) = ax^2 + bx + c$ such that $y_k = f(k)$ for each k. To see why, note that all such $f(x)$ satisfy the given recurrence relation, and we can solve the system of equations

$$\begin{aligned} a + b + c &= y_1, \\ 4a + 2b + c &= y_2, \\ 9a + 3b + c &= y_3 \end{aligned}$$

to find a particular $f(x)$ that also matches the first three terms of our sequence of y_k. Since the entire sequence of y_k is determined by those initial three terms, the entire sequence of y_k matches $f(k)$.

Let $p_m = \sum_{k=1}^{n} k^m$. The condition that $\sum_{k=1}^{n} y_k = 0$ can be rewritten as

$$\sum_{k=1}^{n} (ak^2 + bk + c) = 0,$$

or $a \sum_{k=1}^{n} k^2 + b \sum_{k=1}^{n} k^1 + c \sum_{k=1}^{n} k^0 = 0$. In terms of p_m, this equation is

$$ap_2 + bp_1 + cp_0 = 0.$$

Rewriting the entire first condition in the same way gives the following system of equations:

$$\begin{aligned} ap_2 + bp_1 + cp_0 &= 0, \\ ap_3 + bp_2 + cp_1 &= 0, \\ ap_4 + bp_3 + cp_2 &= 0. \end{aligned}$$

Let $M = \begin{pmatrix} p_2 & p_1 & p_0 \\ p_3 & p_2 & p_1 \\ p_4 & p_3 & p_2 \end{pmatrix}$ and $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Our system of equations states that

$$M \cdot v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve this equation for v, we will try to invert M, which is possible as long as $\det(M) \neq 0$. We have

$$\det(M) = p_4(p_1^2 - p_0p_2) - p_3(p_2p_1 - p_0p_3) + p_2(p_2^2 - p_1p_3).$$

There are well-known formulas for p_0, p_1, p_2, and p_3 and a less-well-known formula for p_4:

$$\begin{aligned} p_0 &= n, \\ p_1 &= \frac{1}{2}n(n+1), \\ p_2 &= \frac{1}{6}n(n+1)(2n+1), \\ p_3 &= \frac{1}{4}n^2(n+1)^2, \\ p_4 &= \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1). \end{aligned}$$

Using these formulas, $\det(M)$ simplifies to

$$\frac{1}{2160}n^3(n+1)^2(n-1)^2(4-n^2),$$

which is negative since $n \geq 4$.

Since the $\det(M) < 0$, M is invertible, so we have

$$v = M^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, we have $a = b = c = 0$, so all the y_k are 0, which implies the desired conclusion.