1/1/28. Fill in each cell of the grid with one of the numbers 1, 2, or 3. After all numbers are filled in, if a row, column, or any diagonal has a number of cells equal to a multiple of 3, then it must have the same amount of 1's, 2's, and 3's. (There are 10 such diagonals, and they are all marked in the grid by a gray dashed line.) Some numbers have been given to you.

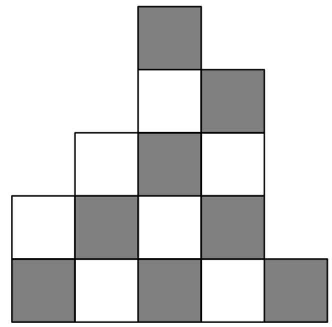
	2	1						
3			2					
			2			3	2	
	2	1			1			3
3					3			3
2						2	3	
3	2	3	2		2			3
					3			1
						1	3	

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

1	2	1	3	2	3	3	1	2
3	3	1	2	3	2	1	2	1
2	3	3	2	1	1	3	2	1
1	2	1	3	2	1	2	3	3
3	1	2	1	2	3	2	1	3
2	1	3	3	1	1	2	3	2
3	2	3	2	1	2	1	1	3
2	1	2	1	3	3	3	2	1
1	3	2	1	3	2	1	3	2

2/1/28. A tower of height h is a stack of contiguous rows of squares of height h such that



(i) the bottom row of the tower has h squares,

(ii) each row above the bottom row has one fewer square than the row below it, and within each row the squares are contiguous,

(iii) the squares in any given row all lie directly above a square in the row below.

A tower is called balanced if when the squares of the tower are colored black and white in a checkerboard fashion, the number of black squares is equal to the number of white squares. For example, the figure above shows a tower of height 5 that is not balanced, since there are 7 white squares and 8 black squares.

How many balanced towers are there of height 2016?

Solution

Each row with an even number of squares has the same number of black and white squares. Each row with an odd number of squares has a number of black and white squares differing by 1. So, a balanced tower must have the same number of odd rows with one more black square as odd rows with one more white square.

In particular, a balanced tower of height 2016 must have 504 odd rows with one more white square and 504 odd rows with one more black square. Given the previous rows, there are two options for each row of the tower, and for odd rows exactly one choice results in one more white square and exactly one choice results in one more black square.

We now obtain the answer by constructive counting. Each even row past the first row can be chosen in two ways, for 2^{1007} choices total. Among the 1008 odd rows, we select 504 row numbers to have one more black square, and the rest to have one more white square. This uniquely determines each row, and there are $\binom{1008}{504}$ ways to select the row numbers. The total number of balanced towers is $\boxed{2^{1007}\dbinom{1008}{504}}$.

3/1/28. Find all positive integers n for which $(x^n + y^n + z^n)/2$ is a perfect square whenever x, y, and z are integers such that $x + y + z = 0$.

Solution

Letting $x = 2, y = -1$, and $z = -1$, we get

$$\frac{x^n + y^n + z^n}{2} = 2^{n-1} + (-1)^n,$$

which must be a square. We deal with the cases where n is even and odd separately.

n *even*: We have

$$2^{n-1} + 1 = m^2$$

for some positive integer m. Subtracting 1 from both sides and factoring gives us

$$2^{n-1} = (m+1)(m-1).$$

This means that $m + 1$ and $m - 1$ must both be powers of 2, so $m = 3$, and hence, $n = 4$.

n *odd*: We have

$$2^{n-1} - 1 = m^2$$

for some nonnegative integer m. Since n is odd, the left-hand side is a difference of squares, so

$$\left(2^{(n-1)/2} + 1\right)\left(2^{(n-1)/2} - 1\right) = m^2.$$

We can see that $n = 1$ is a solution. If $n > 1$, $2^{(n-1)/2} + 1$ and $2^{(n-1)/2} - 1$ are relatively prime and must both be squares. However, no two squares differ by 2, so this is impossible and $n = 1$ is the only solution.

Therefore, the only possibilities are $n = 1$ and $n = 4$. We show that these actually work. If $x + y + z = 0$, then

$$\frac{x + y + z}{2} = 0^2,$$

so $n = 1$ works. For $n = 4$, we have

$$\begin{aligned}\frac{x^4 + y^4 + z^4}{2} &= \frac{x^4 + y^4 + (x+y)^4}{2} \\ &= x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4 \\ &= (x^2 + xy + y^2)^2,\end{aligned}$$

which is a perfect square. Therefore, $n = \boxed{\text{1 and 4}}$ are both solutions.

4/1/28. Find all functions $f(x)$ from nonnegative reals to nonnegative reals such that $f(f(x)) = x^4$ and $f(x) \le Cx^2$ for some constant C.

Solution

One solution is $f(x) = x^2$. We will prove that this is the only solution.

First, since $f(x) \le Cx^2$, we know that $f(0) = 0$. Suppose that for some t, we have $\frac{f(t)}{t^2} = a \neq 1$. Then, using the functional equation for f, we have

$$f(at^2) = f(f(t)) = t^4 = \frac{(at^2)^2}{a^2}.$$

Using the functional equation again gives

$$f(t^4) = f(f(at^2)) = (at^2)^4 = a^4 t^8.$$

Continuing in this manner, we get a sequence $\langle t_n \rangle$ with $t_0 = t$ such that

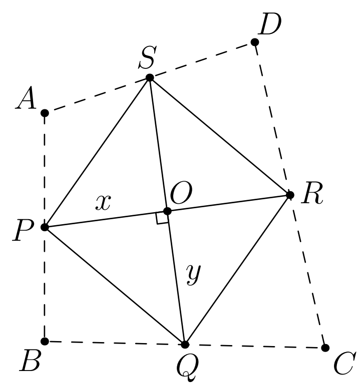
$$f(t_n) = a^{(-2)^n} t_n^2.$$

However, $a^{(-2)^n}$ is unbounded for any $a \neq 0$ or 1. So, for all x, we have either $f(x) = x^2$ or $f(x) = 0$. However, if $f(r) = 0$, then applying f to each side of the equation $r^4 = f(0) = 0$, so $r = 0$. Thus $f(x) = 0$ if and only if $x = 0$.

So, the only solution to the functional equation is $\boxed{f(x) = x^2}$.

5/1/28. Let $ABCD$ be a convex quadrilateral with perimeter $\frac{5}{2}$ and $AC = BD = 1$. Determine the maximum possible area of $ABCD$.

Solution



Let P, Q, R, S denote the midpoints of AB, BC, CD, DA, respectively. Varignon's theorem tells us $PQRS$ is a parallelogram, and we also see that $PQ = \frac{AC}{2}$, etc., so

$$PQ = QR = RS = SP = \frac{1}{2}.$$

Hence $PQRS$ is in fact a rhombus and its diagonals are perpendicular. Suppose its diagonals meet at O.

Using the inequalities $PR \leq \frac{AD+BC}{2}$ and $QS \leq \frac{AB+CD}{2}$, we have

$$PR + QS \leq \frac{AB + BC + CD + DA}{2} = \frac{5}{4}$$

Set $x = PO = \frac{1}{2}PR$ and $y = QO = \frac{1}{2}QS$, noting that the area of $PQRS$ is $2xy$, and that this is half the area of $ABCD$. We know that

$$x + y \leq \frac{5}{8} \text{ and } x^2 + y^2 = PO^2 + QO^2 = PQ^2 = \frac{1}{4}.$$

Therefore,

$$2xy \leq \left(\frac{5}{8}\right)^2 - \frac{1}{4} = \frac{9}{64}$$

so the area of $PQRS$ is at most $\frac{9}{64}$. Thus, the area of $ABCD$ is at most $\boxed{\frac{9}{32}}$.

Note that equality occurs when $ABCD$ is a rectangle with sides $\frac{5+\sqrt{7}}{8}$ and $\frac{5-\sqrt{7}}{8}$.