USA Mathematical Talent Search

Round 2 Solutions

Year 27 — Academic Year 2015–2016

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1/2/27. In the grid to the right, the shortest path through unit squares between the pair of 2's has length 2. Fill in some of the unit squares in the grid so that

(i) exactly half of the squares in each row and column contain a number,

(ii) each of the numbers 1 through 12 appears exactly twice, and

(iii) for $n = 1, 2, \ldots, 12$, the shortest path between the pair of n's has length exactly n.

				6			
	10	1					
			2				8
	5			2			
					7	9	
				3			

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

In this solution, we use the notation R_iC_j to denote the square in row i and column j. For example, the given 5 is in R_4C_2.

The larger numbers give us the strongest restrictions, so we examine them first.

11				6			**12**
	10	1					
			2				8
	5			2			
					7	9	
12				3		**11**	**10**

The only pairs of squares that are a distance of 12 apart are the two pairs of diagonally opposite corners (R_1C_1, R_6C_8) and (R_1C_8, R_6C_1). However, the only square a distance of 10 from the given 10 is R_6C_8. So the 10 must be placed there. Thus, we can place the 12's in the corners (R_1C_8, R_6C_1).

For the pair of 11's to be a distance of 11 apart, one of them must be in a corner. Since three corners are already taken, we place an 11 in the remaining corner, R_1C_1. The two squares that are a distance of 11 from this are R_5C_8 and R_6C_7. Since column 8 already contains 12, 8, and 10, we cannot place any more numbers in that column and we must place the 11 in R_6C_7.

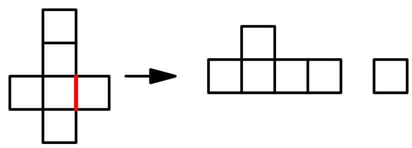
11		**7**		6			12
9	10	1					
			2				8
	5			2			
	8				7	9	
12				3		11	10

Next we resolve the 9, 8, and 7. There are two possible positions for the 7: it can be either in R_3C_1 or R_1C_3. Similarly, the remaining 8 can be placed in R_4C_1, R_6C_3, or R_5C_2 and the remaining 9 can be placed in R_2C_1 or R_1C_2. The 8 cannot be placed in R_6C_3 because row 6 already contains 12, 3, 11, and 10. So all possible positions for the non-given 8 and 9 are in columns 1 and 2. This means that columns 1 and 2 will each contain 3 numbers after the 8 and 9 are placed. So we cannot place the 7 in column 1, as this would force column 1 to contain 4 numbers. So we place the 7 in R_1C_3. This means that row 1 contains four numbers and we must place the 9 in R_2C_1. Once the 9 is placed, column 1 contains three numbers and we must place the 8 in R_5C_2.

Now we are forced to place the 6 in R_5C_3. This forces us to place the 1 in R_2C_4. The only way to place two more numbers in row 3 is to place the 5 in R_3C_6 and the 4 in R_3C_7. We conclude by placing the remaining 4 in R_4C_4 and the remaining 3 in R_4C_6.

11		7		6			12
9	10	1	**1**				
			2		**5**	**4**	8
	5		**4**	2	**3**		
	8	**6**			7	9	
12				3		11	10

2/2/27. A net for a polyhedron is cut along an edge to give two **pieces**. For example, we may cut a cube net along the red edge to form two pieces as shown.



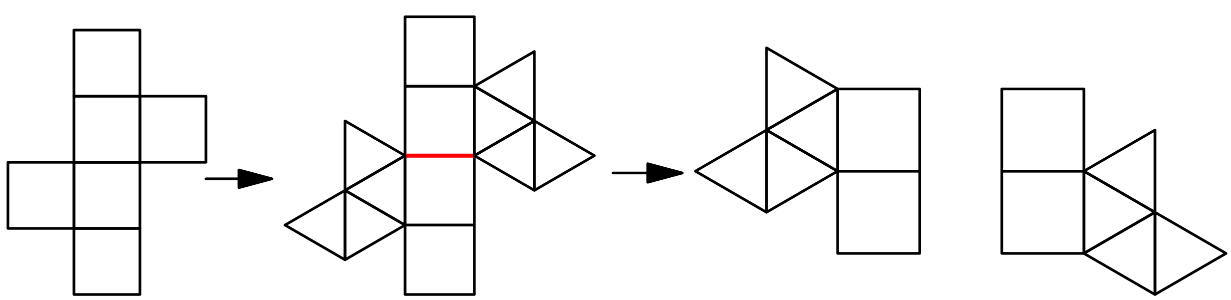
Are there two distinct polyhedra for which this process may result in the same two pairs of pieces?

Solution

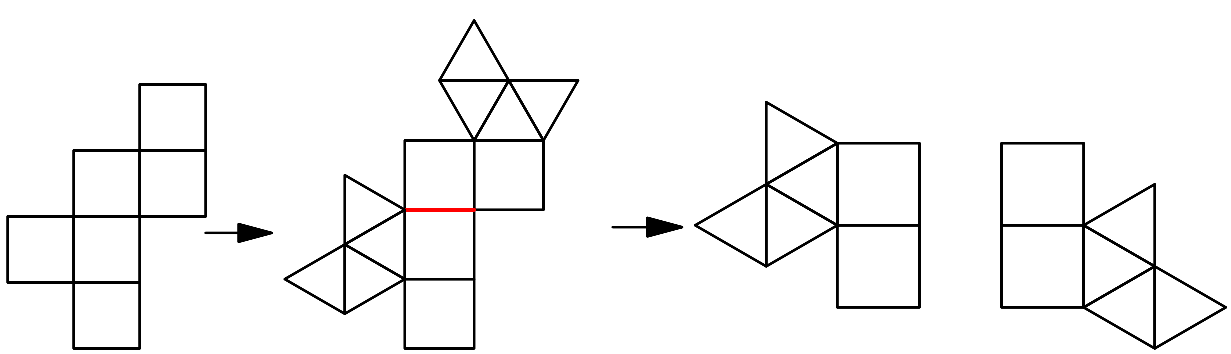
The answer is yes and there are many possible solutions. We present one here.

We start with a cube and glue square pyramids on two of its faces. There are two distinct ways to do this: the two pyramids can replace either adjacent faces or opposite faces. We show that we can find nets for each of these polyhedra that produce the same pair of polygons.

In the diagram below, we replace two opposite faces in the cube net with square pyramids, then cut the resulting net along the red edge to produce two polygons.



In the diagram below, we replace two adjacent faces in the cube net with square pyramids, then cut the resulting net along the red edge to produce two polygons.



3/2/27. For all positive integers n, show that

$$\frac{1}{n}\sum_{k=1}^{n}\frac{k\cdot k!\cdot\binom{n}{k}}{n^k}=1.$$

Solution 1

Fix n. Let

$$S_r=\sum_{k=n-r}^{n}\frac{k\cdot k!\cdot\binom{n}{k}}{n^k}$$

be the sum of the last $r+1$ terms of the given sum. We claim that

$$S_r=\frac{n!}{r!n^{n-r-1}}.$$

For $r=0$, notice that we can write the last term in the sum as

$$\frac{n\cdot n!\cdot\binom{n}{n}}{n^n}=\frac{n\cdot n!}{n^n}=\frac{n!}{0!n^{n-1}}.$$

So the claim holds for $r=0$. Now suppose the claim holds for $r=m-1$ and we will show that it's true for $r=m$ (assuming $m<n$).

The sum of the last $m+1$ terms is the sum of the last m terms, plus the $(m+1)$st term from the end. That is,

$$S_m=S_{m-1}+\frac{(n-m)n!/m!}{n^{n-m}}.$$

By the inductive hypothesis, the sum of the last m terms is

$$S_{m-1}=\frac{n!}{(m-1)!n^{n-m}}.$$

So the sum of the last $m+1$ terms is

$$\begin{aligned}S_m=\frac{n!}{(m-1)!n^{n-m}}+\frac{(n-m)n!/m!}{n^{n-m}}&=\frac{n!(n-m)+n!\cdot m}{(m)!n^{n-m}}\\&=\frac{n!\cdot n}{m!n^{n-m}}\\&=\frac{n!}{m!n^{n-m-1}}.\end{aligned}$$

Therefore, for all $r<n$, the sum of the last $r+1$ terms is $\frac{n!}{r!n^{n-r-1}}$. Substituting $r=n-1$, we have

$$\begin{aligned}\sum_{k=1}^{n}\frac{k\cdot k!\cdot\binom{n}{k}}{n^k}&=\frac{n!}{(n-1)!n^{n-(n-1)-1}}\\&=\frac{n}{n^0}=n.\end{aligned}$$

Dividing through by n gives the result.

Solution 2

Suppose we roll an n-sided die repeatedly until we roll any number for the second time. We calculate the probability that the first repeat is on the $(k+1)$st roll.

In order for the $(k+1)$st roll to be the first repeat, the previous k rolls must have all been distinct. The number of sequences of k distinct rolls is $k!\binom{n}{k}$, because we can choose any k numbers from our die to show up and place them in any order. Since there are n^k total sequences of k rolls, the probability that the first k rolls are distinct is $\frac{k!\binom{n}{k}}{n^k}$. Since the $(k+1)$st roll is a repeat, it must be one of the k numbers already seen. The probability of this happening is $\frac{k}{n}$. In total, the probability that our $(k+1)$st roll is the first repeat is

$$\frac{k}{n} \cdot \frac{k!\binom{n}{k}}{n^k} = \frac{k \cdot k! \cdot \binom{n}{k}}{n^{k+1}}.$$

Summing over all $k \leq n$, we find that the probability that we have at least one repeat after $n+1$ rolls is

$$\sum_{k=1}^{n} \frac{k \cdot k! \cdot \binom{n}{k}}{n^{k+1}}.$$

Since we're rolling an n-sided die, we're guaranteed to have a repeat by the $(n+1)$st roll. In particular,

$$\sum_{k=1}^{n} \frac{k \cdot k! \cdot \binom{n}{k}}{n^{k+1}} = 1.$$

Factoring an n out of the denominator, we have

$$\frac{1}{n} \sum_{k=1}^{n} \frac{k \cdot k! \cdot \binom{n}{k}}{n^{k}} = 1.$$

4/2/27. Find all polynomials $P(x)$ with integer coefficients such that, for all integers a and b, $P(a+b) - P(b)$ is a multiple of $P(a)$.

Solution

Fix an integer b and write $H(x) = P(x+b) - P(b)$. If H is identically 0, then $P(x)$ is a constant function. Substituting $P(x) = c$, the condition says that c divides 0, which is true. So constant functions work. For the remainder, we will assume that $P(x)$ is non-constant. In that case, $H(x)$ and $P(x)$ have the same leading term, so we can write

$$H(x) = P(x) + r(x),$$

where $r(x)$ is either identically 0 or $\deg r(x) < \deg P(x)$. For any integer a, we have that $H(a) = P(a) + r(a)$ is a multiple $P(a)$. In particular, for any integer a, $r(a)$ is a multiple of $P(a)$.

The degree of $r(x)$ is less than the degree of $P(x)$ (or $r(x)$ is identically 0), so we can choose some M such that for all integers $a \geq M$, $|P(a)| > |r(a)|$. Since $r(a)$ is a multiple of $P(a)$ for each such a, this implies that $r(a) = 0$ for all integers $a \geq M$. Therefore, $r(x)$ is a polynomial with infinitely many zeros, and must be identically 0. Hence $H(x) = P(x)$ and

$$P(x+b) - P(b) = P(x).$$

Since b was an arbitrary integer, this equation holds for all integers b and real numbers x. Plugging in $x = b = 0$ yields $P(0) = 0$. Plugging in $x = 1, b = 1$ gives us $P(2) = 2P(1)$. Then plugging in $x = 2, b = 1$ gives us $P(3) = 2P(1) + P(1) = 3P(1)$. Continuing this way by induction, plugging in $x = k, b = 1$ gives us

$$P(k+1) = kP(1) + P(1) = (k+1)P(1).$$

Therefore, $P(x) - xP(1)$ is a polynomial with infinitely many zeros, and must be identically 0. So if $P(x)$ is not constant, $P(x) = xP(1)$ is a linear function with no constant term.

Therefore, the only non-constant solution is $P(x) = cx$ for some constant c. Substituting $P(x) = cx$, the given condition says that $ca + cb - cb = ca$ is a multiple of ca, which is true. So both of these classes of functions work and our solution is $\boxed{P(x) = cx \text{ or } P(x) = c}$ for some integer c.

Note: We ended up deriving an equation of the form $P(x+b) = P(x) + P(b)$. This is a version of Cauchy's functional equation. It turns out that if a function $f : \mathbb{Q} \to \mathbb{Q}$ satisfies $f(x+y) = f(x) + f(y)$ for all x and y, then it must be of the form $f(x) = cx$. The proof is similar to what we did above, except we can no longer take advantage of the fact that we know f is a polynomial.

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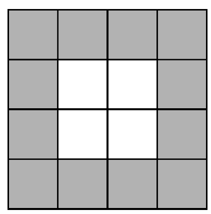
5/2/27. Let $n > 1$ be an even positive integer. A $2n \times 2n$ grid of unit squares is given, and it is partitioned into n^2 contiguous 2×2 blocks of unit squares. A subset S of the unit squares satisfies the following properties:

(i) For any pair of squares A, B in S, there is a sequence of squares in S that starts with A, ends with B, and has any two consecutive elements sharing a side; and

(ii) In each of the 2×2 blocks of squares, at least one of the four squares is in S.

An example for $n = 2$ is shown below, with the squares of S shaded and the four 2×2 blocks of squares outlined in bold.



In terms of n, what is the minimum possible number of elements in S?

Solution

The answer is $\frac{3}{2}n^2 - 2$. Let $n = 2k$ and we will show that the minimum number of elements in S is $6k^2 - 2$. We begin with a Lemma.

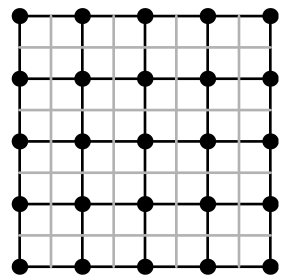
Lemma: Let k be a positive integer. If we have a $(2k+1) \times (2k+1)$ grid of squares, with some squares shaded so that all shaded squares are connected by side, as in condition (i) in the problem, and no 2×2 subgrid is entirely unshaded, then the number of shaded squares is at least $2k^2 - 1$.

Proof: Place the $(2k+1) \times (2k+1)$ grid on the coordinate plane with each square having area 1. The shaded squares together form a polygon whose area is equal to the total number of shaded squares. Since no 2×2 subgrid is unshaded, all of the interior vertices of the grid are inside or on the boundary of the polygon; there are $4k^2$ such vertices. By Pick's Theorem, we have that the area is $\frac{B}{2} + I - 1$ where B is the number of boundary vertices and I is the number of interior vertices. As $B + I \geq 4k^2$, we have

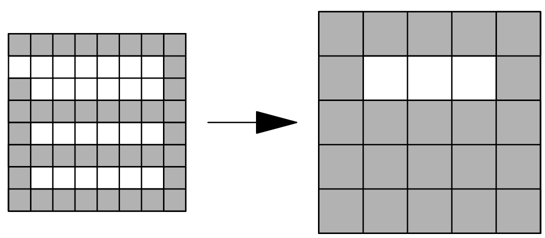
$$\frac{B}{2} + I - 1 \geq \frac{B+I}{2} - 1 \geq \frac{4k^2}{2} - 1 = 2k^2 - 1,$$

as desired. ■

Let S be a valid subset of the original grid, and we will show that it contains at least $6k^2 - 2$ squares. Consider the set of vertices of the n^2 blocks of 2×2 squares.

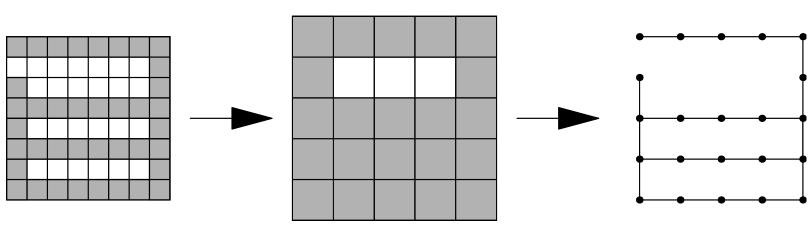


There are $(2k+1)^2$ such vertices. Construct a $(2k+1) \times (2k+1)$ grid G of squares so that each square corresponds to one of these $(2k+1)^2$ vertices. Shade a square in G if the corresponding vertex touches a square of S. An example of such a grid and shading is shown below.



By (i), the shaded squares in G are connected by side, and by (ii) no 2×2 block of squares in G can be fully unshaded. Therefore, by the lemma we have at least $2k^2 - 1$ shaded squares in G.

Next, construct a graph G' as follows: the vertices correspond to the shaded squares in G. Whenever we have two adjacent squares in the same 2×2 block both in S, we draw an edge in G' connecting the two vertices that the two squares in S touch. An example graph is shown below.

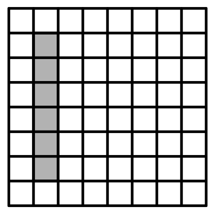


By (i), G' is a connected graph. Let $S(G')$ be a spanning tree of G'. Since G' has at least $2k^2 - 1$ vertices, we know that $S(G')$ has at least $2k^2 - 2$ edges.

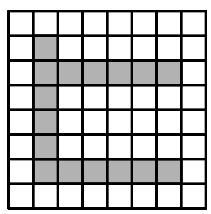
Notice that each edge in $S(G')$ corresponds to two unit squares in the same 2×2 block both being shaded. Since $S(G')$ contains no cycles, each edge corresponds to a unique such pair of unit squares. We use this to place a lower bound on the total number of shaded squares in the original grid.

In the original grid, we require at least one shaded square in each of the $4k^2$ blocks. Also, an additional shaded square in the original grid is required for each edge of $S(G')$. There are at least $2k^2 - 2$ edges in $S(G')$, so in total we have at least $6k^2 - 2$ shaded squares.

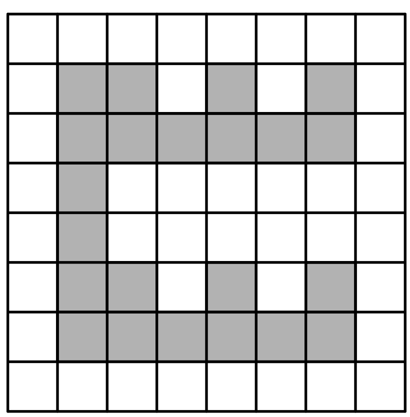
We now exhibit a construction of $6k^2 - 2$ shaded squares. In the accompanying diagrams, we demonstrate the construction for $n = 4$. Number the rows and columns from 1 to $4k$, and denote the square in row i column j as (i, j). Shade in the $4k - 2$ squares $(i, 2)$ for $2 \leq i \leq 4k - 1$.



Next, for the k values of i with $1 \leq i \leq 4k$ and $i \equiv 2 \pmod 4$, shade in $4k - 3$ squares (i, j) for $3 \leq j \leq 4k - 1$.



Finally, shade in $k(2k-1)$ squares of the form $(2i-1, 4j-1)$, for $2 \leq i \leq 2k$ and $1 \leq j \leq k$.



The total number of shaded squares is

$$4k - 2 + k(4k-3) + k(2k-1) = 4k^2 - 3k + 4k - 2 + 2k^2 - k = 6k^2 - 2 = \frac{3}{2}n^2 - 2,$$

as desired.