USA Mathematical Talent Search

Round 1 Solutions

Year 27 — Academic Year 2015–2016

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1/1/27. Fill in the spaces of the grid to the right with positive integers so that in each 2×2 square with top left number a, top right number b, bottom left number c, and bottom right number d, either $a + d = b + c$ or $ad = bc$.

3	9			
	11		7	2
10				16
15				
20	36			32

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

Consider the following 2×2 square

a	b
c	d

and suppose $a < c$. By hypothesis, we must have either $a+d = b+c$ or $ad = bc$. Either way, it follows that $b < d$. Since $9 < 11$, repeatedly applying this observation tells us that every number in the top row is less than the number directly below it. Similarly, every other pair of consecutive rows must satisfy the same constraint, and we conclude that every column is strictly increasing from top to bottom.

Given the same 2×2 configuration, if a positive integer n divides c but not ad, then we must have $a + d = b + c$.

Finally, consider the following 3×2 configuration:

a	e	c
b	f	d

In this case, we claim that if $\gcd(a, b) = 1$ and $b - a > d - c$, then we must have $a + f = b + e$. To see this, let $b = a + k$ and suppose $af = be$. This can be rearranged to $a(f - e) = ke$, or

$$\frac{a}{b-a} = \frac{e}{f-e}.$$

Since a and $b - a$ are relatively prime, we conclude that $f - e \geq b - a$ and $e \geq a$. This implies that we must have $fc = ed$. Rewriting e as $\frac{af}{b}$ yields $ad = bc$, and by the same argument we have $d - c \geq b - a$, a contradiction. So we must have $a + f = b + e$.

Combining these observations with a little trial-and-error, one can construct the following unique solution.

3	9	12	6	1
5	11	14	7	2
10	22	28	21	16
15	27	33	26	21
20	36	44	37	32

2/1/27. Suppose a, b, and c are distinct positive real numbers such that

$$abc = 1000,$$
$$bc(1-a) + a(b+c) = 110.$$

If $a < 1$, show that $10 < c < 100$.

Solution

By adding the two equations we have $ab + ac + bc = 1110$. So a, b, and c are the roots of the polynomial

$$f(x) = (x-a)(x-b)(x-c) = x^3 - dx^2 + 1110x - 1000,$$

where $d > 0$. We compare this to the polynomial

$$g(x) = (x-1)(x-10)(x-100) = x^3 - 111x^2 + 1110x - 1000.$$

Since $a < 1$, we know that $g(a) < 0$. Notice that $g(x) - f(x) = (d-111)x^2$. Since $g(a) = g(a) - f(a) < 0$, we conclude that $g(x) - f(x) = (d-111)x^2$ is negative for all $x \neq 0$. Therefore,

$$g(b) = g(b) - f(b) < 0$$

and

$$g(c) = g(c) - f(c) < 0.$$

This means that b and c are in $(0,1) \cup (10,100)$. Since $abc = 1000$, we see that $bc > 1000$, which implies that b and c are both greater than 10. Thus, $10 < b, c < 100$, as desired.

Note: This proof can be generalized. Given any two cubic polynomials differing in exactly one coefficient and having six distinct positive roots, one can show that there are only two possible orderings of the six roots.

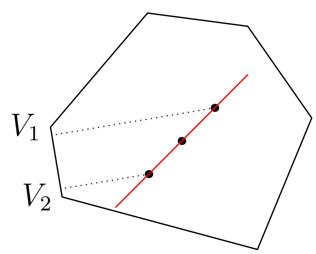
3/1/27. Let P be a convex n-gon in the plane with vertices labeled $V_1, \ldots, V_n$ in counterclockwise order. A point Q not outside P is called a *balancing point* of P if, when the triangles $QV_1V_2, QV_2V_3, \ldots, QV_{n-1}V_n, QV_nV_1$ are alternately colored blue and green, the total areas of the blue and green regions are the same. Suppose P has exactly one balancing point. Show that the balancing point must be a vertex of P.

Solution

Define a function $f : P \to \mathbb{R}$ by setting

$$f(Q) = [QV_1V_2] - [QV_2V_3] + \cdots + (-1)^{n-1}[QV_nV_1].$$

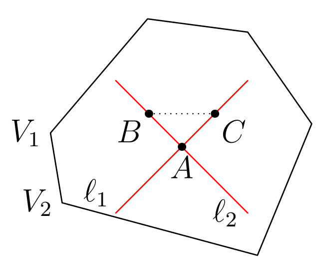
That is, $f(Q)$ is the area of the blue regions minus the area of the green regions given a fixed point Q. We examine what happens to f as we move Q along a fixed line. To do this, we look at the area of the triangle with base V_1V_2.



The base stays fixed and the height changes linearly as Q moves along this line. The same is true for all other sides of P, which means that $f(Q)$ changes linearly as Q moves along a fixed line inside P.

Now suppose A is not a vertex and $f(A) = 0$. We'll show that P contains at least one more balancing point. In order to show this, we will consider two cases: (1) A is in the interior of P and (2) A is on an edge of P.

Suppose first that A is in the interior of P. Take two segments ℓ_1 and ℓ_2 containing A in their interiors inside P. If either segment contains another balancing point, we're done. So suppose neither does. Since f changes linearly as we move along either segment, we know that we can find two points B and C on ℓ_1 and ℓ_2, respectively, such that $f(B)$ and $f(C)$ have opposite signs.

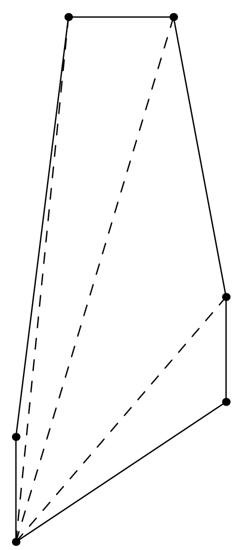


By convexity, we know that the segment connecting B and C is completely contained in P. Since f changes linearly along this segment, there must be some D between B and C for which $f(D) = 0$. Therefore, we have found another balancing point.

Now suppose that A is on an edge, V_iV_{i+1}. Pick some point R inside P and consider the segment QR. Again, if either V_iV_{i+1} or QR contains another balancing point we're done, so suppose neither does. Since f changes linearly as we move along V_iV_{i+1} and QR, we know that we can find two points B' and C' on V_iV_{i+1} and QR, respectively, such that $f(B')$ and $f(C')$ have opposite signs. By convexity, we know that the segment connecting B' and C' is completely contained in P. Since f changes linearly on this segment, there must be some D' between B' and C' for which $f(D') = 0$, and once again we have found another balancing point.

Combining both of these cases, we see that if P has exactly one balancing point, it must be a vertex.

Note: A hexagon with vertices at $(0, 0), (0, 2), (1, 10), (3, 10), (4, 14/3)$, and $(4, 8/3)$ has exactly one balancing point at $x = 0$.



One can calculate the areas of the four triangles (from left to right) to be 1, 10, 13, and 4, and we see that $(0, 0)$ is indeed a balancing point.

4/1/27. Several players try out for the USAMTS basketball team, and they all have integer heights and weights when measured in centimeters and pounds, respectively. In addition, they all weigh less in pounds than they are tall in centimeters. All of the players weigh at least 190 pounds and are at most 197 centimeters tall, and there is exactly one player with every possible height-weight combination.

The USAMTS wants to field a competitive team, so there are some strict requirements.

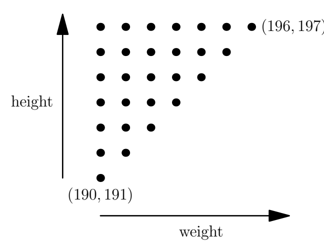
(i) If person P is on the team, then anyone who is at least as tall and at most as heavy as P must also be on the team.

(ii) If person P is on the team, then no one whose weight is the same as P's height can also be on the team.

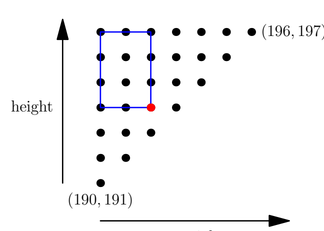
Assuming the USAMTS team can have any number of members (including zero), how many distinct basketball teams can be constructed?

Solution

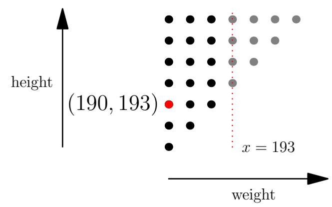
We interpret players as points in the plane with coordinates $(x, y) = (\text{weight}, \text{height})$ and interpret the requirements graphically. Each player is a lattice point on or within the right triangle defined by the lines $y = x + 1$, $x = 190$, and $y = 197$.



Requirement (i) states that if a player (a, b) is on the team, then so is everyone on or within the rectangle above and to the left of (a, b). An example for player $(192, 194)$ is shown below.

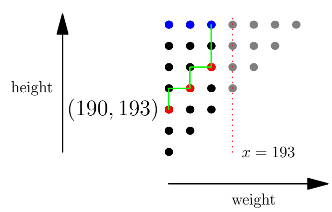


Requirement (ii) states that if a player (a, b) is on the team, then no one on the line $x = b$ can be on the team. Combined with requirement (i), this additionally implies that no one to the right of the line $x = b$ can be on the team if a player (a, b) is on the team. An example for player $(190, 193)$ is shown below.



Notice that the red line is determined by the shortest player selected for the team on the line $x = 190$. Call this player p_0, and let his height be $190 + m$. Then requirement (ii) tells us that all players selected for the team are on the lines $x = 190, x = 191, \dots, x = 190+(m-1)$. Requirement (i) implies that the players selected for the team on the line $x = 190 + k$ are determined by the shortest player selected for the team on this line. Let p_k be the shortest player on the line $x = 190 + k$. Then it suffices to count the number of ways to select $p_0, \dots, p_{m-1}$. (If there are no players selected for the team on the line $x = 190+k$, we simply ignore $p_k, \dots, p_{m-1}$.)

To do this, draw a path containing only steps up and to the right starting at p_0 and ending at one of the m dots in the top row to the left of the line $x = 190 + m$, such that the lowest point on the path with x-coordinate $190 + k$ is p_k (if there are no players with x-coordinate $190 + k$, our path will stay completely to the left of the line $x = 190 + k$). An example path is shown below with $p_0 = (190, 193), p_1 = (191, 194)$, and $p_2 = (192, 195)$.



These paths correspond to teams because they determine $p_0, \dots, p_{m-1}$. Notice that these paths are in one to one correspondence with paths starting at p_0 that end at $(189 + m, 198)$: we simply remove the portion of any given path above the line $y = 197$. Each path from our initial player to $(189 + m, 198)$ must have $m - 1$ steps to the right and $7 - (m - 1)$ steps up. Thus the total number of paths from our initial player to $(189 + m, 198)$ is $\binom{7}{m-1}$. Summing over all $m \leq 7$, we get

$$\sum_{m=1}^{7} \binom{7}{m-1} = 2^7 - 1$$

paths. This corresponds to the number of teams with at least one player, so in total the number of possible teams is $2^7 = 128$.

Challenge: Can you give a bijective proof mapping possible teams to binary strings of 0's and 1's of length 7?

5/1/27. Find all positive integers n that have distinct positive divisors $d_1, d_2, \ldots, d_k$, where $k > 1$, that are in arithmetic progression and

$$n = d_1 + d_2 + \cdots + d_k.$$

Note that $d_1, \ldots, d_k$ do not need to be all the divisors of n.

Solution

$n = 6$ works by taking $d_1 = 1$, $d_2 = 2$, and $d_3 = 3$. Similarly, $n = 6m$ works by taking $d_1 = m$, $d_2 = 2m$, and $d_3 = 3m$. We will show that these are the only possible values of n.

We'll assume without loss of generality that $d_1 < d_2 < \cdots < d_k$. Then, by dividing out by any common factor, we'll assume that $d_1, d_2, \ldots, d_k$ share no (non-trivial) common divisor, making our goal to show that $n = 6$.

Suppose two consecutive terms d_i and d_{i+1} have a common divisor r. Then $r \,|\, (d_{i+1} - d_i)$, which is the common difference of the arithmetic sequence. This implies that r divides all d_i, but we assumed that $d_1, d_2, \ldots, d_k$ shared no common divisor, so $r = 1$.

Since d_k and d_{k-1} are relatively prime and are both factors of n, we have $n \geq d_k d_{k-1}$. Then, $d_k \geq k$, so $n \geq k d_{k-1}$. For $k \geq 3$, d_{k-1} is greater than or equal to the average of the d_i, which means that n is at least k times the average of d_i. That is,

$$n \geq d_1 + d_2 + \cdots + d_k.$$

In order for equality to hold, we need $n = k d_{k-1} = d_k d_{k-1}$. The first equality implies that d_{k-1} is the average of the k-term arithmetic sequence, so $k = 3$. The second equality then tells us that $d_3 = 3$. That is, $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, and $n = 6$, as desired.

To conclude it now suffices to eliminate $k = 2$. In this case, we have $n = d_1 + d_2$ and $d_1 \neq d_2$, which means that $d_2 > \frac{n}{2}$, contradicting the fact that d_2 is a divisor of n. Thus $k = 2$ is impossible.