USA Mathematical Talent Search
Round 3 Solutions
Year 26 — Academic Year 2014–2015
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1/3/26. Fill in each blank unshaded cell with a positive integer less than 100, such that every consecutive group of unshaded cells within a row or column is an arithmetic sequence.

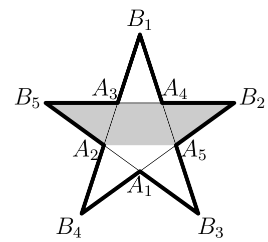
3							59			
	10									
				31	26					59

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

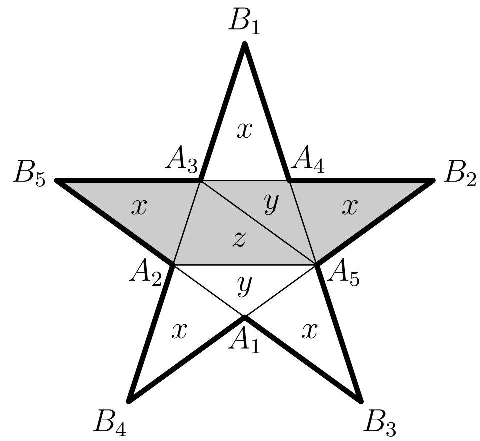
3	6	9	12	15	18		59	71	83	95
		13	18		20	35	50	65		
3	10	17	24		22		41	59	77	95
27			30		24		32			77
51	46	41	36	31	26		23	35	47	59

2/3/26. Let $A_1A_2A_3A_4A_5$ be a regular pentagon with side length 1. The sides of the pentagon are extended to form the 10-sided polygon shown in bold at right. Find the ratio of the area of quadrilateral $A_2A_5B_2B_5$ (shaded in the picture to the right) to the area of the entire 10-sided polygon.



Solution

Decompose the pentagon into 8 triangles as shown below with areas x, y, and z (by symmetry, $A_3A_4A_5 \cong A_5A_1A_2$, so these two triangles have the same area, y). Note that 4 triangles are shaded and 4 are white.



The interior angles of the pentagon are 108°, so

$$\angle A_4A_3A_5 = \frac{1}{2}(180^\circ - 108^\circ) = 36^\circ,$$

and thus

$$\angle A_2A_3A_5 = 108^\circ - 36^\circ = 72^\circ.$$

From this, we see that the triangle with area z is isosceles with base angles 72°. Each of the outer triangles has base angle equal to the exterior angle of a pentagon, so these are also isosceles with base angle 72°. Since z and x are areas of similar triangles that share a base, these triangles are congruent and $x = z$.

The white area is then $3x + y$ and the shaded area is also $2x + y + z = 3x + y$. Therefore the shaded area is exactly $\frac{1}{2}$ of the entire area.

3/3/26. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers such that:

(i) For all positive integers m and n, we have $a_{mn} = a_m a_n$, and

(ii) There exists a positive real number B such that for all positive integers m and n with $m < n$, we have $a_m < Ba_n$.

Find all possible values of $\log_{2015}(a_{2015}) - \log_{2014}(a_{2014})$.

Solution

Condition (i) implies that $a_{n^k} = (a_n)^k$ for all positive integers n and k, by induction on k. Hence,

$$(a_n)^k = a_{n^k} < Ba_{(n+1)^k} = B(a_{n+1})^k,$$

and thus $\left(\frac{a_n}{a_{n+1}}\right)^k < B$. As k can be arbitrarily large, we must have $\frac{a_n}{a_{n+1}} \le 1$, or $a_n \le a_{n+1}$. That is, $a_1, a_2, a_3, \ldots$ is a nondecreasing sequence. Also note that condition (i) with $m = 1$ gives $a_n = a_1 a_n$, so $a_1 = 1$.

If $a_2 = 1$, then $a_{2^k} = 1^k = 1$ for all positive integers k, and hence (because the a's are nondecreasing) we must have $a_n = 1$ for all n. In this case, $\log_{2015}(1) - \log_{2014}(1) = 0 - 0 = 0$.

If $a_2 = \lambda > 1$, then we claim that $\log_n(a_n) = \log_2(\lambda)$ for all $n \ge 2$. To prove the claim, let p be any positive integer such that $2^p > n$, and then set q to be the unique positive integer such that

$$n^q < 2^p \le n^{q+1}.$$

Since the a's are nondecreasing, $a_{n^q} \le a_{2^p} \le a_{n^{q+1}}$. Thus, $(a_n)^q \le (a_2)^p \le (a_n)^{q+1}$. Because $a_2 = \lambda$,

$$(a_n)^q \le \lambda^p \le (a_n)^{q+1}.$$

Taking the logarithm base 2 of both inequality chains gives

$$\begin{array}{rcccl} q\log_2(n) & < & p & \le & (q+1)\log_2(n), \\ q\log_2(a_n) & \le & p\log_2(\lambda) & \le & (q+1)\log_2(a_n). \end{array}$$

Since all of these logarithms are positive, we can divide the bottom inequality chain by the top one (in reverse order) to get

$$\frac{q\log_2(a_n)}{(q+1)\log_2(n)} \le \log_2(\lambda) \le \frac{(q+1)\log_2(a_n)}{q\log_2(n)}.$$

This simplifies to

$$\left(\frac{q}{q+1}\right)\log_n(a_n) \le \log_2(\lambda) \le \left(\frac{q+1}{q}\right)\log_n(a_n).$$

But as p grows, q will grow as well, and both $\frac{q}{q+1}$ and $\frac{q+1}{q}$ can be made arbitrarily close to 1. Therefore, $\log_n(a_n) = \log_2(\lambda)$, proving the claim.

Hence, in this case, $\log_{2015}(a_{2015}) - \log_{2014}(a_{2014}) = \log_2(\lambda) - \log_2(\lambda) = 0$.

Therefore, the only possible value of $\log_{2015}(a_{2015}) - \log_{2014}(a_{2014})$ is 0.

4/3/26. Nine distinct positive integers are arranged in a circle such that the product of any two non-adjacent numbers in the circle is a multiple of n and the product of any two adjacent numbers in the circle is not a multiple of n, where n is a fixed positive integer. Find the smallest possible value for n.

Solution

For any positive integer m and prime p, let $\nu_p(m)$ be the exponent of the largest power of p that divides m. In other words, $\nu_p(m)$ is the exponent of p in the prime factorization of m. Note also that the function ν_p satisfies $\nu_p(ab) = \nu_p(a) + \nu_p(b)$ for any positive integers a, b.

Assume that n satisfies the condition above with the numbers on the circle being $x_1, x_2, \dots, x_9$. Set $x_{10} = x_1$ and $x_{11} = x_2$ (to simplify the notation). For each $1 \le i \le 9$, since n does not divide $x_i x_{i+1}$, there must be a prime p such that $\nu_p(x_i x_{i+1}) < \nu_p(n)$. Let p_i be one such prime, so $\nu_{p_i}(x_i) + \nu_{p_i}(x_{i+1}) < \nu_{p_i}(n)$.

Suppose that $p_i = p_j$ for two nonconsecutive indices i, j (where 1 and 9 are treated as consecutive indices). Note that $x_i, x_{i+1}, x_j, x_{j+1}$ are all distinct. Let $p = p_i = p_j$ (to make the notation simpler), and set $\nu_p(n) = k$. Then we have $\nu_p(x_i) + \nu_p(x_{i+1}) < k$ and $\nu_p(x_j) + \nu_p(x_{j+1}) < k$. Summing, this gives

$$\nu_p(x_i) + \nu_p(x_{i+1}) + \nu_p(x_j) + \nu_p(x_{j+1}) < 2k. \tag{1}$$

On the other hand, x_i and x_j are nonadjacent, so $x_i x_j$ is a multiple of n, and hence $\nu_p(x_i) + \nu_p(x_j) \ge k$. Similarly, x_{i+1} and x_{j+1} are nonadjacent, so $\nu_p(x_{i+1}) + \nu_p(x_{j+1}) \ge k$. Summing, this gives

$$\nu_p(x_i) + \nu_p(x_j) + \nu_p(x_{i+1}) + \nu_p(x_{j+1}) \ge 2k. \tag{2}$$

(2) contradicts (1), so we cannot have $p_i = p_j$ for two nonconsecutive indices i, j.

Now suppose that $p_i = p_{i+1}$. Again, set $p = p_i = p_{i+1}$ to simplify the notation. If $\nu_p(n) = 1$, then neither $x_i x_{i+1}$ nor $x_{i+1} x_{i+2}$ is a multiple of p, so none of x_i, x_{i+1}, x_{i+2} is a multiple of p. But $x_i x_{i+2}$ must be a multiple of n, and hence a multiple of p, a contradiction. Therefore, $\nu_p(n) \ge 2$, and hence p^2 divides n.

To summarize: a prime can appear at most twice in the sequence $p_1, p_2, \dots, p_9$. All primes $p_1, \dots, p_9$ must divide n, since $\nu_{p_i}(n) > 0$ by definition. If any prime appears twice, we know that the square of that prime divides n. So, we may conclude that $p_1 p_2 \cdots p_9$ divides n. We minimize this product by choosing the four smallest primes twice each and the fifth smallest prime once. This gives $n \ge p_1 p_2 \cdots p_9 \ge 2^2 3^2 5^2 7^2 11^1$, so the minimum possible n is $n = 2^2 3^2 5^2 7^2 11^1 = 485100$. This is achievable by using these numbers in order:

$$\frac{n}{2 \cdot 11}, \frac{n}{2^2}, \frac{n}{2 \cdot 3}, \frac{n}{3^2}, \frac{n}{3 \cdot 5}, \frac{n}{5^2}, \frac{n}{5 \cdot 7}, \frac{n}{7^2}, \frac{n}{7 \cdot 11}.$$

5/3/26. A finite set S of unit squares is chosen out of a large grid of unit squares. The squares of S are tiled with isosceles right triangles of hypotenuse 2 so that the triangles do not overlap each other, do not extend past S, and all of S is fully covered by the triangles. Additionally, the hypotenuse of each triangle lies along a grid line, and the vertices of the triangles lie at the corners of the squares. Show that the number of triangles must be a multiple of 4.

Solution

Think of the hypotenuse of each triangular tile as a perfectly reflecting mirror and the legs as transparent glass. From the midpoint of the hypotenuse of one of the tiles, shine a laser toward the midpoint of either leg of the tile. The unit square containing the leg must be completely covered by tiles, so a second tile lies on the other side of this leg, and the laser will strike the midpoint of that tile's hypotenuse and be reflected. Therefore, the laser will be reflected continually, and it will first retrace its path when it returns to the tile it started in. We will show the number of unit squares the laser passed through on this loop must be a multiple of 4.

Let a be the number of squares in which the laser travelled northwest, b be the number of squares in which the laser travelled northeast, c be the number of squares in which the laser travelled southwest, and d be the number of squares in which the laser travelled southeast. Since the laser returned to where it started, it travelled north as many times as it did south. Thus, $a + b = c + d$. Likewise, it travelled west as many times as it did east, so $a + c = b + d$. Finally, the laser alternated between travelling northwest or southeast and travelling northeast or southwest each time it was reflected. So there is a 1-1 correspondence between the two pairs of move types, and $a + d = b + c$. That is, we have

$$a + b = c + d, \qquad a + c = b + d, \qquad a + d = b + c.$$

Adding the first two equations gives $2a = 2d$ so $a = d$. Likewise adding the first and third equation gives $a = c$ and the final two equations gives $a = b$ so $a = b = c = d$. The total number of squares the laser passed through is $a + b + c + d$, which is $4a$ and thus a multiple of 4.

To complete the proof, place a laser in one of the tiles as specified above, and remove all the tiles it passes through from the board. This removes whole unit squares only, and the number of unit squares removed is a multiple of 4. Repeat this process: placing a laser in a tile and removing all the tiles it passes through. Eventually we must remove all the tiles, and since the number of tiles removed for each laser was a multiple of 4, the total number of tiles was also a multiple of 4, as desired.