USA Mathematical Talent Search

Round 2 Solutions

Year 26 — Academic Year 2014–2015

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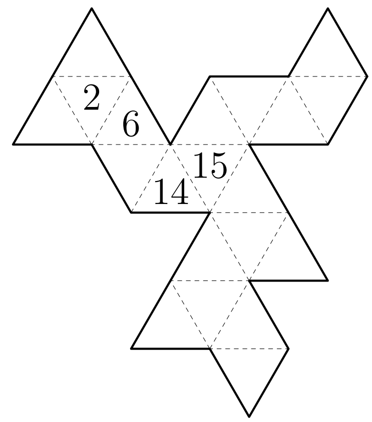
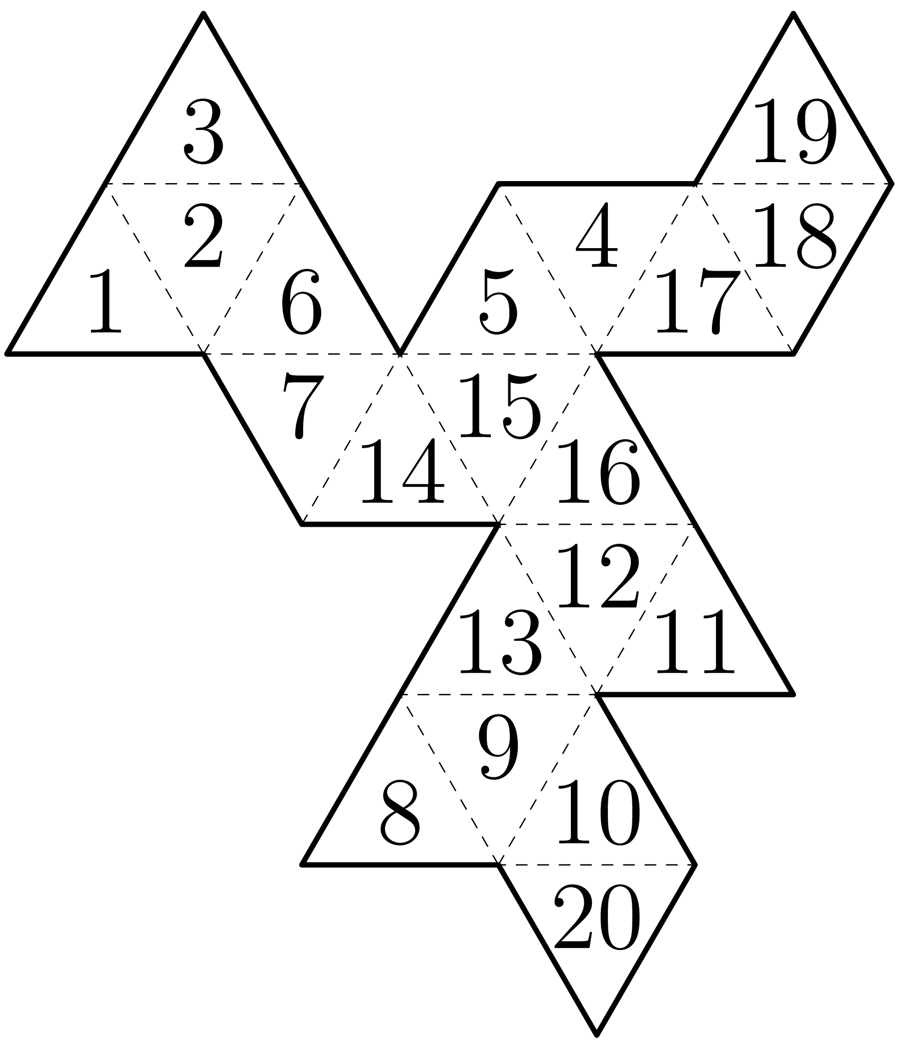
1/2/26. The net of 20 triangles shown to the right can be folded to form a regular icosahedron. Inside each of the triangular faces, write a number from 1 to 20 with each number used exactly once. Any pair of numbers that are consecutive must be written on faces sharing an edge in the folded icosahedron, and additionally, 1 and 20 must also be on faces sharing an edge. Some numbers have been given to you.



You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution



2/2/26. Let a, b, c, x, y be positive real numbers such that

$$ax + by \leq bx + cy \leq cx + ay.$$

Prove that $b \leq c$.

Solution

The first inequality is equivalent to

$$(b - c)y \leq (b - a)x,$$

and the second inequality is equivalent to

$$(b - c)x \leq (a - c)y.$$

If $b > c$, then the left sides of the two inequalities above are positive, so the right sides are positive as well. In particular, this means that $b - a > 0$ and $a - c > 0$. But then $b > a > c$, giving $ax > cx$ and $by > ay$. Adding these last two inequalities gives $ax+by > cx+ay$, which contradicts the relation between the first and third expressions in the original inequality chain.

Thus $b \leq c$.

3/2/26. Let $\mathcal{P}$ be a square pyramid whose base consists of the four vertices $(0,0,0)$, $(3,0,0)$, $(3,3,0)$, and $(0,3,0)$, and whose apex is the point $(1,1,3)$. Let $\mathcal{Q}$ be a square pyramid whose base is the same as the base of $\mathcal{P}$, and whose apex is the point $(2,2,3)$. Find the volume of the intersection of the interiors of $\mathcal{P}$ and $\mathcal{Q}$.

Solution

Consider the cross-sections of $\mathcal{P}$ and $\mathcal{Q}$ given by the plane $z=c$ for some $0 \le c \le 3$.
The cross-section of $\mathcal{P}$ with $z=c$ is the square with vertices:

$$\left(\frac{c}{3},\frac{c}{3},c\right),\left(3-\frac{2c}{3},\frac{c}{3},c\right),\left(3-\frac{2c}{3},3-\frac{2c}{3},c\right),\left(\frac{c}{3},3-\frac{2c}{3},c\right).$$

The cross-section of $\mathcal{Q}$ with $z=c$ is the square with vertices:

$$\left(\frac{2c}{3},\frac{2c}{3},c\right),\left(3-\frac{c}{3},\frac{2c}{3},c\right),\left(3-\frac{c}{3},3-\frac{c}{3},c\right),\left(\frac{2c}{3},3-\frac{c}{3},c\right).$$

Thus the point (x,y,z) is in the interior of both $\mathcal{P}$ and $\mathcal{Q}$ if and only if:

$$\begin{aligned} \frac{z}{3} < x < 3-\frac{2z}{3} \quad &\text{and} \quad \frac{2z}{3} < x < 3-\frac{z}{3}, \\ \frac{z}{3} < y < 3-\frac{2z}{3} \quad &\text{and} \quad \frac{2z}{3} < y < 3-\frac{z}{3}. \end{aligned}$$

Combining these conditions gives

$$\frac{2z}{3} < x < 3-\frac{2z}{3} \quad \text{and} \quad \frac{2z}{3} < y < 3-\frac{2z}{3}.$$

This is equivalent to (x,y,z) lying in the interior of a pyramid with the same base as $\mathcal{P}$ and $\mathcal{Q}$, but with apex $\left(\frac{3}{2},\frac{3}{2},\frac{9}{4}\right)$. (We note that the two inequalities above only have solutions for $z<\frac{9}{4}$, and are symmetric in x and y.) This pyramid has a base of area 9 and a height of $\frac{9}{4}$, so its volume is

$$\frac{1}{3}bh = \frac{1}{3}(9)\left(\frac{9}{4}\right) = \frac{27}{4}.$$

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4/2/26. A point P in the interior of a convex polyhedron in Euclidean space is called a *pivot point* of the polyhedron if every line through P contains exactly 0 or 2 vertices of the polyhedron. Determine, with proof, the maximum number of pivot points that a polyhedron can contain.

Solution

A polyhedron can contain 1 pivot point: for example, the center of a cube is a pivot point.

However, we claim that a convex polyhedron cannot contain more than one pivot point.

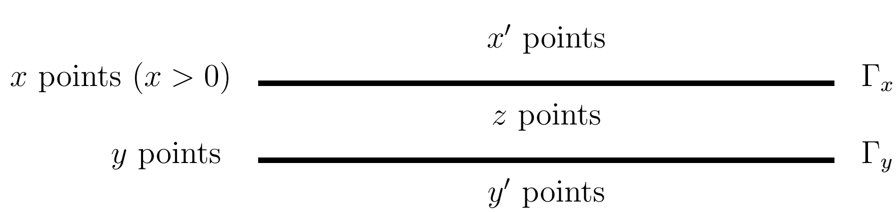
First, we prove the following:

Lemma: Let X be a pivot point of a convex polyhedron $\mathcal{P}$, and let Γ be any plane containing X. Then there are an equal number of vertices in the two half-spaces determined by Γ.

Proof: Let f be a mapping from the set of vertices of $\mathcal{P}$ to itself, defined by setting $f(A)$ to be the unique vertex of $\mathcal{P}$, other than A, on the line $\overleftrightarrow{AX}$. (This vertex must exist, and be uniquely determined, by the definition of a pivot point.) Note that f is its own inverse, so it is a bijection of the vertices of $\mathcal{P}$.

If A is a vertex of $\mathcal{P}$, then the points between X and A are all interior to $\mathcal{P}$ by convexity. Hence, $f(A)$ cannot lie between X and A. Symmetrically, A cannot lie between X and $f(A)$. Therefore X lies between A and $f(A)$. Thus the segment $\overline{Af(A)}$ must either lie on or cross Γ. Therefore, f establishes a 1-1 correspondence between the vertices in one half-space and the vertices in the other half-space, and thus there are an equal number of them. $\square$

Now we can prove our claim. Suppose, for the sake of contradiction, that there exists a convex polyhedron $\mathcal{P}$ that contains two distinct pivot points X and Y. Choose distinct parallel planes Γ_X through X and Γ_Y through Y such that Γ_X contains at least one vertex of $\mathcal{P}$. Let x and y be the number of vertices of $\mathcal{P}$ on Γ_X and Γ_Y, respectively, and note that by construction $x > 0$. Also let z be the number of vertices of $\mathcal{P}$ that lie between Γ_X and Γ_Y, and let x' and y' (respectively) be the number of vertices in the half-space lying on the side of Γ_X (respectively Γ_Y) that does not contain Γ_Y (respectively Γ_X). The diagram below shows a side view of the planes Γ_X and Γ_Y, along with the number of points on, between, and to either side of the planes.



Then by the Lemma, we have

$$x' = z + y + y',$$
$$y' = z + x + x'.$$

Adding these equations and canceling $x' + y'$ from both sides gives

$$0 = 2z + x + y.$$

However, x is positive and z and y are nonnegative, giving the contradiction.
Thus, a convex polyhedron can contain at most 1 pivot point.

5/2/26. Find the smallest positive integer n that satisfies the following: We can color each positive integer with one of n colors such that the equation

$$w + 6x = 2y + 3z$$

has no solutions in positive integers with all of w, x, y, z the same color. (Note that w, x, y, z need not be distinct: for example, 5 and 7 must be different colors because $(w, x, y, z) = (5, 5, 7, 7)$ is a solution to the equation.)

Solution

The minimum number of colors is 4.

First, we show that 4 colors is achievable. We color every positive integer with one of four colors according to its base-3 representation, as follows:

Number of terminating 0's in base-3 representation	Right-most nonzero digit in base-3 representation	Color
even	1	red
even	2	blue
odd	1	green
odd	2	yellow

We show that, using the above coloring, there are no solutions of our equation in which all four variables are of the same color. Assume this coloring does admit a solution. If there is a blue solution (w, x, y, z), then the quadruple $(2w, 2x, 2y, 2z)$ is red and is also a solution. If there exists a green solution (w, x, y, z), then $(3w, 3x, 3y, 3z)$ is a red solution. If there exists a yellow solution (w, x, y, z) then $(6w, 6x, 6y, 6z)$ is a red solution. Therefore if there is any solution in this coloring then there is a red solution. We need only prove that this coloring admits no red solutions.

If we are given a red solution (w, x, y, z), then each variable ends in one of $00, 01, 11$, or 21 in base 3, and so each variable is congruent to 0, 1, 4, or 7 $\pmod 9$. The values that $w + 6x$ can take on are summarized in the following table (with values of w along the top and values of x along the side):

	0	1	4	7
0	0	1	4	7
1	6	7	1	4
4	6	7	1	4
7	6	7	1	4

The values that $2y + 3z$ can take on are summarized in the following table (with values of y along the top and values of z along the side):

	0	1	4	7
0	0	2	8	5
1	3	5	2	8
4	3	5	2	8
7	3	5	2	8

Therefore $w + 6x$ is congruent to 0, 1, 4, 6, or 7 $\pmod 9$. Likewise, $2y + 3z$ is congruent to 0, 2, 3, 5, or 8 $\pmod 9$. The only way to satisfy $w + 6x = 2y + 3z$ is for both to be congruent to 0, meaning all four of (w, x, y, z) are multiples of 9.

The contradiction is quick from here. Let (w, x, y, z) be some red solution that minimizes w. Since each variable is a multiple of 9, dividing by 9 gives us a new solution which is still red (because we have removed an even number of terminating 0s), contradicting the minimality of w.

Now we show that coloring the positive integers using 3 (or fewer) colors is insufficient. Note that for any positive integer k, the 4-tuple $(2k, k, k, 2k)$ is a solution to the equation, as are the 4-tuples $(3k, k, 3k, k)$ and $(3k, 2k, 3k, 3k)$. Thus, k, $2k$ and $3k$ must all be different colors, which shows that at least 3 colors are necessary.

Suppose we have only 3 colors (red, blue, green). Without loss of generality, since $\{1, 2, 3\}$ must all be different colors, we can color 1 red, 2 blue, and 3 green. Then since $\{2, 4, 6\}$ and $\{3, 6, 9\}$ must each be all different colors, 6 must be a different color than both 2 and 3, so 6 must be red, which makes 4 green and 9 blue.

Below is a chart showing the colors so far:

1	2	3	4	5	6	7	8	9
R	B	G	G		R			B

Next, note that $\{4, 8, 12\}$ and $\{6, 12, 18\}$ must each be all different colors. In particular, 12 must be a different color than both 4 and 6, so 12 must be blue. Then, 8 must be different than both 4 and 12, so 8 must be red.

1	2	3	4	5	6	7	8	9	10	11	12
R	B	G	G		R		R	B			B

But now, we are not able to color 5, because:

- If 5 is red, then $(1, 5, 8, 5)$ is an all-red solution to the equation.
- If 5 is green, then $(5, 3, 4, 5)$ is an all-green solution to the equation.
- If 5 is blue, then $(9, 5, 12, 5)$ is an all-blue solution to the equation.

Therefore, we cannot color the integers using just 3 (or fewer) colors.