USA Mathematical Talent Search

Round 1 Solutions

Year 26 — Academic Year 2014–2015

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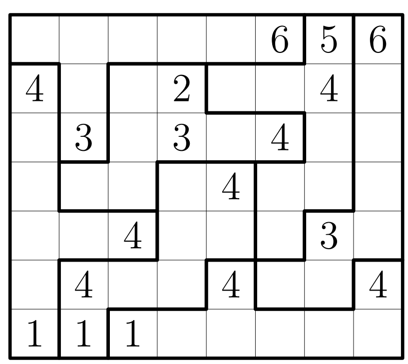
1/1/26. Divide the grid shown to the right into more than one region so that the following rules are satisfied.

					6	5	6
4			2			4	
	3		3		4		
				4			
		4				3	
	4			4			4
1	1	1					

1. Each unit square lies entirely within exactly 1 region.
2. Each region is a single piece connected by the edges of its unit squares.
3. Each region contains the same number of whole unit squares.
4. Each region contains the same sum of numbers.

Solution

Let there be n regions, and let a be the area of each region and s be the sum of the numbers in each region. Observe that the area of the entire grid is 56 and that the sum of all of the numbers in the grid is 63. Therefore, we have $na = 56$ and $ns = 63$. But we also know that n, a, s are integers and $n > 1$. Therefore, n must be a nontrivial common divisor of 56 and 63. The only such number is 7, so therefore $n = 7$, which gives $a = 8$ and $s = 9$. Thus, there will be 7 regions, each region will contain 8 squares, and each region's numbers will sum to 9.



Next, observe that no region can contain any two of the 6s and 5 in the top right, since its sum would be too high. Additionally, no region can contain both a number at least 5 and a 1, since the regions have area 8 and these numbers are too far away from each other. Therefore, the only way to form a region containing a 6 is to have it contain 6 and 3. Similarly, the only way to make a region containing a 5 is to have it contain a 5 and a 4. So two regions contain the numbers 3,6 and one other contains 4,5; the remaining numbers in the grid are 1,1,1,2,3, and seven 4s. Among these numbers, the only way to form a sum of 9 containing a 1 is 1,4,4. Since there are three 1s, three regions contain the numbers 1,4,4. The final region contains the numbers 2,3,4. So, we have that the seven regions contain these sets of numbers:

$$1, 4, 4; \quad 1, 4, 4; \quad 1, 4, 4; \quad 2, 3, 4; \quad 3, 6; \quad 3, 6; \quad 4, 5.$$

From here, it is fairly straightforward by trial-and-error to construct the following partition of the grid into the 7 desired regions, as shown at right above.

2/1/26. Find all triples (x, y, z) such that x, y, z, $x-y$, $y-z$, $x-z$ are all prime positive integers.

Solution

Note that since $x-y$, $y-z$, $x-z$ are positive, we must have $x > y > z$.

We cannot have more than one of x, y, z be even, since there is only one even prime. But if x, y, z are all odd, then $x-y$ and $x-z$ are distinct even primes, a contradiction. Therefore exactly one of x, y, z is an even prime, and since 2 is the smallest prime, we must have $z = 2$.

Thus x and y are both odd. But then $x-y$ is even and prime, so $x-y=2$ and hence $x = y+2$. Therefore our triple is $(y+2, y, 2)$. This means that all of $y+2$, y, and $y-2$ are prime. But at least one of these is a multiple of 3, and the only multiple of 3 that is prime is 3. The only possibility is $y-2=3$, giving $y=5$ and $y+2=7$.

Therefore, the only such triple is $(x, y, z) = \boxed{(7, 5, 2)}$.

3/1/26. A group of people is lined up in *almost-order* if, whenever person A is to the left of person B in the line, A is not more than 8 centimeters taller than B. For example, five people with heights 160, 165, 170, 175, and 180 centimeters could line up in almost-order with heights (from left-to-right) of 160, 170, 165, 180, 175 centimeters.

(a) How many different ways are there to line up 10 people in almost-order if their heights are 140, 145, 150, 155, 160, 165, 170, 175, 180, and 185 centimeters?

(b) How many different ways are there to line up 20 people in almost-order if their heights are 120, 125, 130, 135, 140, 145, 150, 155, 160, 164, 165, 170, 175, 180, 185, 190, 195, 200, 205, and 210 centimeters? (Note that there is someone of height 164 centimeters.)

Solution

(a) We prove the following Lemma:

Lemma: If we have n people whose heights are $a, a+5, \ldots, a+(n-1)5$ centimeters for some a, then there are F_{n+1} ways to line them up in almost-order, where F_k is the k^{th} Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for all $k \geq 2$.

Proof of Lemma: We prove by induction on n. The base cases $n = 1$ and $n = 2$ have $F_2 = 1$ and $F_3 = 2$ almost-orderings, respectively.

For the inductive step, let $n > 2$ be given and assume that the result holds for fewer than n people. Let L be a valid almost-ordered line up of the n people.

If the tallest person is at the rightmost end of L, then the preceding $n-1$ people are almost-ordered, so by the inductive hypothesis there are F_n such almost-orderings.

If the tallest person is not at the rightmost end of L, then he must be no more than 5 centimeters taller than any person to his right. But there is only 1 such person: the next-tallest person who is 5 centimeters shorter than the tallest person. So we must have the next-tallest person at the far right end of L, and the tallest person to his immediate left. This means that the preceding $n-2$ people are almost-ordered (and they are all shorter than the two people at the far-right end), so by the inductive hypothesis there are F_{n-1} such almost-orderings.

Combining both cases, we get $F_n + F_{n-1} = F_{n+1}$ total ways to almost-order a group of n people, completing the induction. $\square$

Applying this Lemma to the given set of 10 people gives $F_{11} = \boxed{89}$ possible almost-orderings.

(b) First, we note that only the people of heights 160 or 170 can be lined up between the people of heights 164 and 165. There are two cases. If the person of height 164 is to the left of the person with height 165, then by the almost-ordering condition, if someone of height h is between them, we have $164 < h + 9$ and $h < 165 + 9$. Therefore $155 < h < 174$, so h is either 160 or 170. In the case that the person of height 164 is to the right of the person with height 165, we obtain $165 < h + 9$ and $h < 164 + 9$, so again h must be either 160 or 170.

We now consider four cases based on the number of people who are between the people of heights 164 and 165.

Case 1: *No person appears between the people of heights 164 and 165.* By the Lemma, there are $F_{20} = 6765$ almost-orderings of the 19 people of heights 140, 145, ..., 205, 210 (excluding the person of height 164). For any almost-order of these 19 people, the person of height 164 can be added directly to the left or the directly to the right of the person of height 165. Thus, the number of almost-orderings including the person of height 164 is twice the number of almost-orderings of the 19 people of heights 140, 145, ..., 205, 210. There are $2 \times 6765 = 13,530$ total almost-orderings in this case.

Case 2: *Only the person of height 160 is between the people of heights 164 and 165.* Suppose that the person of height 164 is to the left of the person of height 165. Then all the people of height 155 or less must be to the left of the person of height 164, and all the people of height 170 or more must be to the right of the person of height 165. By the Lemma, there are $F_9 = 34$ ways to almost-order the 8 people on the left, and $F_{10} = 55$ ways to almost-order the 9 people on the right. This yields $34 \times 55 = 1870$ almost-orderings. Since an identical analysis applies if the person of height 165 is to the left of the person of height 164, there are $2 \times 1870 = 3740$ almost-orderings in this case.

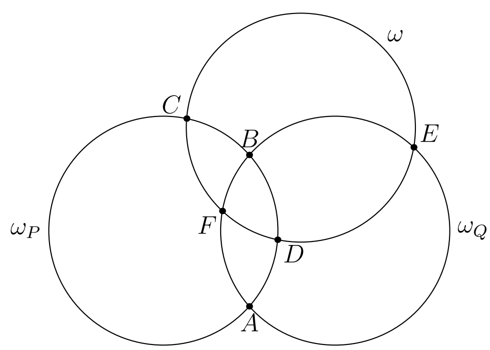
Case 3: *Only the person of height 170 is between the people of heights 164 and 165.* Suppose that the person of height 164 is to the left of the person of height 165. Then all the people of height 160 or less must be to the left of the person of height 164, and all the people of height 175 or more must be to the right of the person of height 165. By the Lemma, there are $F_{10} = 55$ ways to almost-order the 9 people on the left, and $F_9 = 34$ ways to almost-order the 8 people on the right. This yields $55 \times 34 = 1870$ almost-orderings. Since an identical analysis applies if the person of height 165 is to the left of the person of height 164, there are $2 \times 1870 = 3740$ almost-orderings in this case.

Case 4: *Both the person of height 160 and the person of height 170 are between the people of heights 164 and 165.* Because of the almost-ordering condition, the person of height 160 must be to the left of the person of height 170. Thus, the only two almost-orderings of these four people are 164, 160, 170, 165 and 165, 160, 170, 164. Suppose that the person of height 164 is to the left of the person of height 165. Then all people of height 155 or less must be to the left of the person of height 164, and all people of height 175 or more must be to the right of the person of height 165. By the Lemma, there are $F_9 = 34$ ways to almost-order the eight people on the left, and $F_9 = 34$ ways to almost-order the 8 people on the right. This yields $34 \times 34 = 1156$ almost-orderings. Since an identical analysis applies if the person of height 165 is to the left of the person of height 164, there are $2 \times 1156 = 2312$ total almost-orderings in this case.

Adding up all the cases gives us $13,530 + 3740 + 3740 + 2312 = \boxed{23,322}$ total almost-orderings of the 20 people.

4/1/26. Let ω_P and ω_Q be two circles of radius 1, intersecting in points A and B. Let P and Q be two regular n-gons (for some positive integer $n \geq 4$) inscribed in ω_P and ω_Q, respectively, such that A and B are vertices of both P and Q. Suppose a third circle ω of radius 1 intersects P at two of its vertices C, D and intersects Q at two of its vertices E, F. Further assume that A, B, C, D, E, F are all distinct points, that A lies outside of ω, and that B lies inside ω. Show that there exists a regular $2n$-gon that contains C, D, E, F as four of its vertices.

Solution



Note that the statement of the problem asks for the existence of a $2n$-gon whenever the construction is complete. Although the construction is not possible for all n (for example, it cannot be done for $n = 4$ or $n = 5$), the result does hold whenever the construction can be completed as follows.

Since B is inside ω, the three circles have nontrivial intersection, so one of C, D is inside ω_Q and one of E, F is inside ω_P. Without loss of generality, suppose that D is inside ω_Q and F is inside ω_P, as shown in the picture to the right. This means that the order of points on ω is C, F, D, E.

Let $s = \frac{2\pi}{n}$, the length of the arc between any two adjacent points in the regular n-gon. Note that the fact that A, B, C, D are vertices of a regular n-gon means that arcs $\widehat{AD}$, $\widehat{DB}$, and $\widehat{BC}$ all have lengths that are positive integer multiples of s, and similarly for arcs $\widehat{AF}$, $\widehat{FB}$, and $\widehat{BE}$.

Let x, y, z be the lengths of arcs $\widehat{CF}$, $\widehat{FD}$, and $\widehat{DE}$, respectively. To prove our result, it suffices to show that x, y, and z are each half-integer multiples of s, because in a $2n$-gon the length of the arc between any two adjacent points is $s/2$.

Note that there is an arc $\widehat{CD}$ in both ω_P and ω, and since the circles have the same radius, this arc has the same length in both circles. In circle ω_P, we know that this arc length is a multiple of s, so it is also a multiple of s in circle ω, and thus $x + y$ is an integer multiple of s. Similarly, by looking at arc $\widehat{EF}$ in circles ω_Q and ω, we see that $y + z$ is an integer multiple of s.

Next, since the angles of triangle BDF add up to π, we have $\angle BDF + \angle DFB + \angle FBD = \pi$. But $\angle BDF$ is inscribed in arcs $\widehat{FC}$ and $\widehat{BC}$, so in terms of arc lengths, $\widehat{FC} + \widehat{CB} = 2\angle BDF$. Making similar computations with the other two angles in BDF gives us

$$\widehat{FC} + \widehat{CB} + \widehat{BE} + \widehat{ED} + \widehat{DA} + \widehat{AF} = 2\pi.$$

But $\widehat{FC} + \widehat{DE} = x + z$, and the other four arcs are all integer multiples of s. Therefore since $2\pi = ns$, we have that $x + z$ is an integer multiple of s as well.

Finally, observe that

$$x = \frac{(x+y)+(x+z)-(y+z)}{2},$$

so since the numerator is an integer multiple of s, we conclude that x is a half-integer multiple of s. Similar computations show that y and z are each half-integer multiples of s. This proves our result.

5/1/26. Let $a_0, a_1, a_2, \ldots$ be a sequence of nonnegative integers such that $a_2 = 5$, $a_{2014} = 2015$, and $a_n = a_{a_{n-1}}$ for all positive integers n. Find all possible values of a_{2015}.

Solution

Let $a_3 = x$ and $a_4 = y$ for some nonnegative integers x and y. Note that

$$x = a_3 = a_{a_2} = a_5 = a_{a_4} = a_y, \tag{1}$$

and that

$$y = a_4 = a_{a_3} = a_x. \tag{2}$$

Claim: For all $n \geq 3$, $a_n = \begin{cases} x & \text{if } n \text{ is odd,} \\ y & \text{if } n \text{ is even.} \end{cases}$

Proof of Claim: We prove by induction. The base cases $n = 3$ and $n = 4$ are true by definition.

Let $n > 4$ be given, and assume the claim is true for all a_k with $3 \leq k < n$. If n is odd, then $a_n = a_{a_{n-1}} = a_y = x$ (by (1)). If n is even, then $a_n = a_{a_{n-1}} = a_x = y$ (by (2)). Therefore the claim is true for a_n, and hence for all $n \geq 3$ by induction. □

Since $a_{2014} = 2015$, we must have $y = 2015$. Thus the sequence is

$$a_0, a_1, 5, x, 2015, x, 2015, x, \ldots$$

with $a_x = 2015$ and $a_{2015} = x$.

Next, we must have $5 = a_2 = a_{a_1}$, which leads to four possibilities:

Case 1: a_1 is an odd integer greater than 1 and $x = 5$. But then $a_x = a_5 = x = 5$, contradicting the fact that $a_x = 2015$. So this case cannot happen.

Case 2: a_1 is an even integer greater than 2. But then $a_{a_1} = 2015$, a contradiction. So this case cannot happen.

Case 3: $a_1 = 2$. Now $2 = a_1 = a_{a_0}$, which gives two subcases:

Subcase a: $a_0 = 1$, giving the sequence

$$1, 2, 5, x, 2015, x, 2015, x, \ldots.$$

The condition $a_x = 2015$ forces x to be either 2015 or an even integer greater than 2.

Subcase b: a_0 is an odd integer greater than 1, and $x = 2$. But then we must have $a_2 = 2015$, which cannot occur since $a_2 = 5$. So this subcase cannot happen.

Case 4: $a_1 = 0$ and $a_0 = 5$. This satisfies $a_2 = a_{a_1} = a_0 = 5$. However, $a_1 = a_{a_0} = a_5 = x$ forces $x = 0$, which contradicts $a_x = 2015$ since $a_0 = 5$. Thus, this case cannot happen.

Thus, $a_{2015} = x$ can be $\boxed{\text{2015 or any even integer greater than 2}}$.