USA Mathematical Talent Search

Round 3 Solutions

Year 25 — Academic Year 2013–2014

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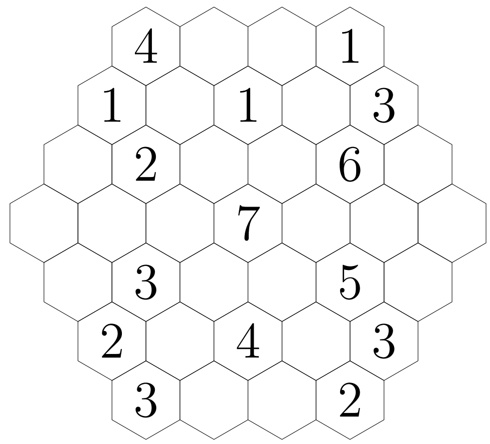
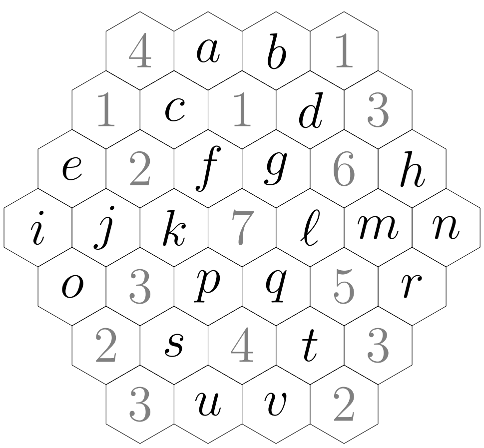
1/3/25. In the hexagonal grid shown, fill in each space with a number. After the grid is completely filled in, the number in each space must be equal to the smallest positive integer that does not appear in any of the touching spaces.



You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

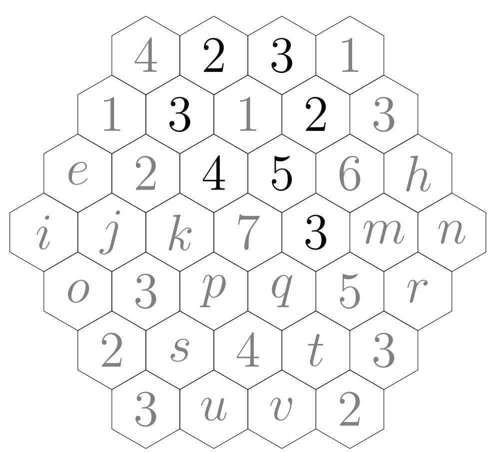


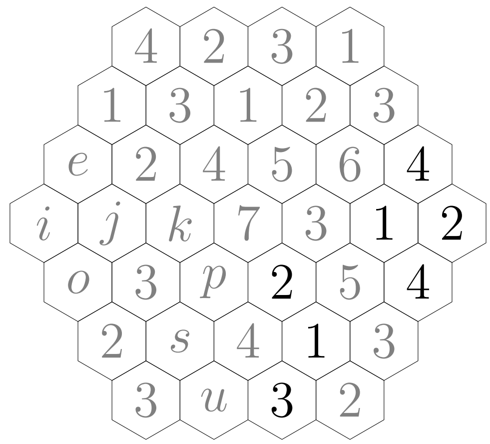
We label the cells a through v lexicographically as in the diagram to the left.

The 4 in row 1 must touch a 2, but the 2 can't be c because c already touches a 2. Therefore, $a = 2$. The 4 must also touch a 3, so $c = 3$. Finally, b must be at least 3, but can't be greater than 3 because there is no way to place a 3 in a neighboring space. Therefore, $b = 3$.

The 6 in the third row must touch a 5. That 5 cannot be ℓ or m because those each neighbor the 5 in the fifth row. The 5 also cannot be d or h because there are not enough empty spaces for d or h to neighbor all of 1, 2, 3, and 4. Therefore, $g = 5$. Now the three vacant spaces touching g must have the numbers 2, 3, and 4. Since f touches a 2 and a 3, it must be the 4. Since d touches a 3, it must be the 2. This leaves $\ell = 3$. The progress so far is summarized in the diagram to the right.

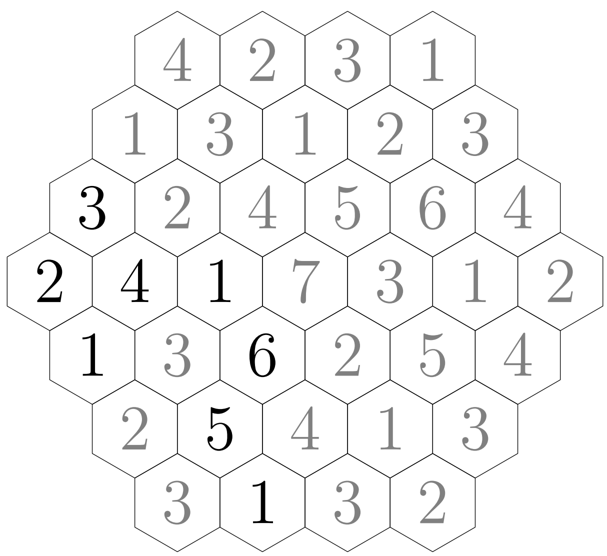
Consider the 4 in the sixth row. It must touch a 3, but the only space surrounding it that does not touch a 3 is v, so $v = 3$. The 2 in row 7 must touch a 1, so $t = 1$. Next, consider $\ell = 3$. It must touch a 1 and a 2, so m and q must have a 1 and 2 between them. But q touches a 1, so $q = 2$ and $m = 1$. To finish the right edge, consider the 5 in the fifth row. It must touch a 4, so $r = 4$. We can quickly determine that $n = 2$ and $h = 4$ from here.





Our progress so far is summarized in the diagram to the right. Next, the 7 in the center must touch the six numbers 1 to 6 once each. We already found the numbers 2 through 5, so k and p are 1 and 6 between them. Notice that regardless of which of the two is the 6, the remaining vacant space it touches (either j or s) must be a 5. In particular, this means each of j and s is either 5 or neighbors a 1, so neither can be a 1. Since the 3 in row 7 must touch a 1, that 1 must be u. Then $s = 5$, since it now touches 1 through 4, but $p \neq 5$.

The 2 in row 6 must touch a 1, so $o = 1$. All of the vacant spaces around i touch a 2, so i is at most 2. It is also at least 2 since it touches a 1, so $i = 2$. By similar reasoning, e is a 3 and j is a 4. Since j is not a 5, the 6 must be p, and the 1 goes in k. This completes the grid. The only solution is the one shown below.



2/3/25. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers such that $a_k a_{k+2} = a_{k+1} + 1$ for all positive integers k. If a_1 and a_2 are both positive integers, find the maximum possible value of a_{2014}.

Solution

First we use the recurrence relation $a_k a_{k+2} = a_{k+1} + 1$ to compute a_{2014} in terms of a_1 and a_2, then we'll choose values of a_1 and a_2 that maximize a_{2014}.

The recurrence relation can be rewritten as

$$a_{k+2} = \frac{a_{k+1} + 1}{a_k}.$$

Using this, we compute

$$\begin{aligned}
a_3 &= \frac{a_2 + 1}{a_1} \\
a_4 &= \frac{a_3 + 1}{a_2} = \frac{\frac{a_2+1}{a_1} + 1}{a_2} = \frac{a_2 + a_1 + 1}{a_1 a_2}, \\
a_5 &= \frac{a_4 + 1}{a_3} = \frac{\frac{a_2+a_1+1}{a_1 a_2} + 1}{\frac{a_2+1}{a_1}} = \frac{(a_1 a_2 + a_2 + a_1 + 1)(a_1)}{(a_1 a_2)(a_2 + 1)} = \frac{(a_1 + 1)(a_2 + 1)(a_1)}{(a_1 a_2)(a_2 + 1)} = \frac{a_1 + 1}{a_2}, \\
a_6 &= \frac{a_5 + 1}{a_4} = \frac{\frac{a_1+1}{a_2} + 1}{\frac{a_2+a_1+1}{a_1 a_2}} = \frac{\frac{a_1+a_2+1}{a_2}}{\frac{a_1+a_2+1}{a_1 a_2}} = a_1, \\
a_7 &= \frac{a_6 + 1}{a_5} = \frac{a_1 + 1}{\frac{a_1+1}{a_2}} = a_2.
\end{aligned}$$

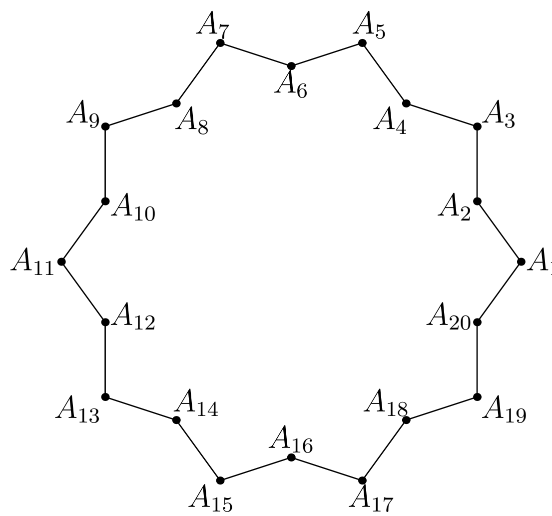
Since each term of the sequence depends only on the previous two terms and we have shown $a_1 = a_6$ and $a_2 = a_7$, we conclude that the sequence is periodic with period 5. Therefore, for any k, we have $a_k = a_r$, where r is the remainder when k is divided by 5. So

$$a_{2014} = a_4 = \frac{a_2 + a_1 + 1}{a_1 a_2} = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_1 a_2}.$$

We are given that a_1 and a_2 are positive integers, so each term in this expression is at most 1 and the sum is at most 3. For $a_1 = a_2 = 1$, the sum is equal to 3. The maximum possible value of a_{2014} is 3 from the sequence

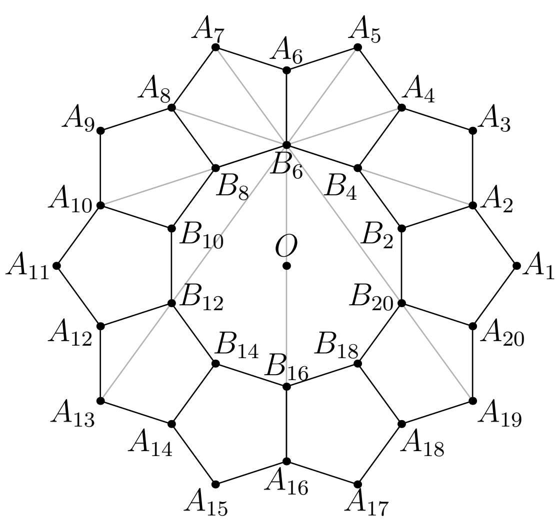
$$1, 1, 2, 3, 2, 1, 1, 2, 3, 2, \ldots.$$

3/3/25. Let $A_1A_2A_3\dots A_{20}$ be a 20-sided polygon P in the plane. Suppose all of the side lengths of P are 1, the interior angle at A_i measures 108 degrees for all odd i, and the interior angle at A_i measures 216 degrees for all even i. Prove that that the lines $A_1A_9, A_2A_{12}, A_3A_{15}, A_4A_{18}$, and A_6A_{20} all intersect at the same point.



Solution

Let O be the center of P. Let B_{2i} be the point on the line segment $\overline{A_{2i}O}$ such that $\overline{A_{2i}B_{2i}}$ has length 1. For each i, draw segment $\overline{A_{2i}B_{2i}}$ and $\overline{B_{2i}B_{2i+2}}$. The resulting figure is shown below left, with the line segments $\overline{A_2A_8}, \overline{A_4A_{10}}, \overline{A_5A_{13}}, \overline{A_6A_{16}}$, and $\overline{A_7A_{19}}$ in gray.



We claim that the 10 small pentagons are regular unit pentagons. By symmetry we may check only $A_2A_3A_4B_4B_2$, so consider only this pentagon for now. First note that $\overline{A_2B_2}$ bisects a 216° angle so $\angle A_2 = 108°$. Since $\angle A_2 = \angle A_3 = \angle A_4 = 108°$ and $\angle B_4 = \angle B_2$, the pentagon is equiangular. By construction, 4 edges have length 1, so all 5 edges have length 1 and the pentagon is regular. In particular, since the pentagon is regular, the line segments $\overline{B_{2i}B_{2i-2}}$ have length 1. By symmetry, the inner decagon is also equiangular so it must be regular. Specifically, each angle of the decagon has measure 144°.

We now claim that $A_2A_8, A_4A_{10}, A_5A_{13}, A_6A_{16}$, and A_7A_{19} all intersect at B_6.

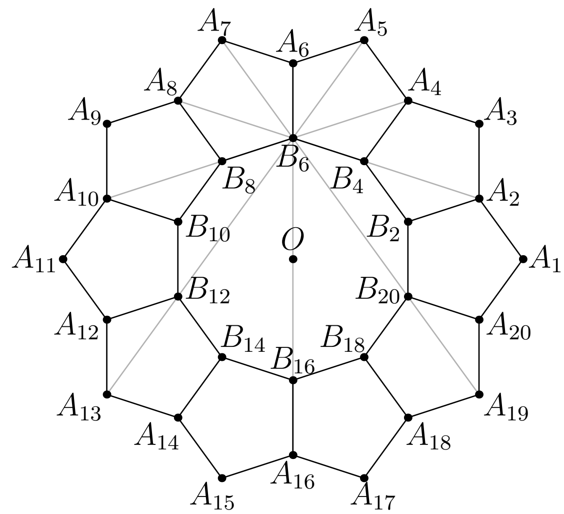
Since B_6 lies on A_6O by definition, A_6A_{16} necessarily passes through B_6. If A_2A_8 passes through B_6, then A_4A_{10} passes through B_6, because A_4A_{10} is the reflection of A_2A_8 through A_6A_{16}. Similarly, if A_5A_{13} passes through B_6, then so does A_7A_{19}. So to solve the problem, it suffices to show that A_2A_8 and A_5A_{13} both contain B_6.

Lines B_4B_6 and A_2A_8 are both perpendicular to A_5A_{15} so are parallel (and hopefully equal). The point B_6 lies on A_2A_8 if and only if these lines are equal. This occurs if and only if $\angle B_4B_6A_8 = 180°$. To prove this, we first write $\angle B_4B_6A_8 = \angle B_4B_6B_8 + \angle B_8B_6A_8$. Since $\angle B_4B_6B_8$ is the interior angle of a regular decagon, its measure is 144°. Further, $\triangle B_6B_8A_8$ is isosceles with vertex angle 108°, so the base angle is $\angle B_8B_6A_8 = 36°$. This forces

$$\angle B_4B_6A_8 = 144° + 36° = 180°,$$

and A_2A_8 passes through B_6.

Similarly, B_6B_{12} is parallel to A_5A_{13}, so A_5A_{13} contains B_6 if and only if $B_6B_{12}A_{13} = 180°$. To prove this, first write



$$\begin{aligned}\angle B_6B_{12}A_{13} = \angle B_6B_{12}B_{10} + \angle B_{10}B_{12}A_{12} \\ + \angle A_{12}B_{12}A_{13}.\end{aligned}$$

Since $\angle A_{12}B_{12}A_{13}$ is the smaller angle formed by a diagonal and a side in a regular pentagon, its measure is $36°$. Since $\angle B_{10}B_{12}A_{12}$ is an interior angle of a regular pentagon, its measure is $108°$. Therefore

$$\begin{aligned}\angle B_6B_{12}A_{13} &= \angle B_6B_{12}B_{10} + 108° + 36° \\ &= \angle B_6B_{12}B_{10} + 144°.\end{aligned}$$

To find the measure of $\angle B_6B_{12}B_{10}$, we view it as an interior angle of the quadrilateral $B_6B_8B_{10}B_{12}$. The sum of the interior angles of a quadrilateral is $360°$. Two of the interior angles of the quadrilateral are interior angles of a regular decagon, and therefore have measure $144°$. The other two angles are also equal to one another, since they are reflections of each other about the line A_9A_{19}. So we have

$$144° + 144° + 2\angle B_6B_{12}B_{10} = 360°.$$

Dividing by 2 gives

$$\angle B_6B_{12}A_{13} = \angle B_6B_{12}B_{10} + 144° = 180°.$$

therefore A_5A_{13} passes through B_6. All 5 lines meet at B_6.

4/3/25. An infinite sequence $(a_0, a_1, a_2, \ldots)$ of positive integers is called a *ribbon* if the sum of any eight consecutive terms is at most 16; that is, for all $i \geq 0$,

$$a_i + a_{i+1} + \cdots + a_{i+7} \leq 16.$$

A positive integer m is called a *cut size* if every ribbon contains a set of consecutive elements that sum to m; that is, given any ribbon $(a_0, a_1, a_2, \ldots)$, there exist nonnegative integers $k \leq \ell$ such that

$$\sum_{i=k}^{\ell} a_i = m.$$

Find, with proof, all cut sizes, or prove that none exist.

Solution

We claim that m is a cut size if and only if m is a positive multiple of 16. First we show that any positive integer m that is not a multiple of 16 is not a cut size. We do this by constructing counter-examples based on the highest power of 2 that divides m.

If m is odd, then m is not a sum of terms from the ribbon

$$2, 2, 2, 2, 2, 2, 2, 2, \ldots.$$

Thus any cut size is even. If m is an odd multiple of 2 (meaning $m \equiv 2 \pmod 4$) then m is not a sum of terms from the ribbon

$$3, 1, 3, 1, 3, 1, 3, 1, \ldots.$$

Specifically, any sum of an even number of consecutive terms from this sequence is a multiple of 4 and any odd number of terms gives an odd sum. Therefore any cut size is a multiple of 4. Likewise, the ribbon

$$5, 1, 1, 1, 5, 1, 1, 1, \ldots$$

shows that any cut size must be a multiple of 8 and the period 8 ribbon

$$9, 1, 1, 1, 1, 1, 1, 1, \ldots$$

shows that any cut size must be a multiple of 16.

Now we must show that for any m that is a multiple of 16, every ribbon contains a consecutive set of terms that sums to m. Let some $m = 16n$ be given.

Consider first all ribbons $a_0, a_1, \ldots$ such that each set of 8 consecutive terms gives a sum of 16:

$$a_i + a_{i+1} + \cdots + a_{i+8} = 16.$$

We call such a ribbon **maximal**. Given a maximal ribbon, the sum of the first $8n$ terms is

$$a_0 + a_1 + \cdots + a_{8n} = 16n = m$$

showing that m can be obtained as a sum of consecutive terms from this ribbon. Now we are left to show that m can be achieved as a sum of consecutive terms for any ribbon that is not maximal.

For any i let $s_i = a_0 + a_1 + \cdots + a_i$ be the sum of the first $i+1$ terms. Since the a_i are all positive integers, s_i is a strictly increasing sequence of positive integers. We wish to find i, j such that $s_j - s_i = m$, which means that $a_{i+1} + a_{i+2} + \cdots + a_j = m$.

Since our ribbon is not maximal, there exists some k such that $s_{k+8} - s_k < 16$. Consider the set $T = \{s_k, 1 + s_k, \ldots 15 + s_k\}$. Since $s_{k+8} < 16 + s_k$, the nine partial sums $s_i, s_{i+1}, \ldots, s_{i+8}$ are members of T. Next consider $U = \{m + s_k, m + 1 + s_k, \ldots m + 15 + s_k\}$. We will show that at least 8 of these values are also partial sums. Let s_b be the largest partial sum smaller than $s_k + m$. This means s_{b+1} is an element of U. Since

$$s_{b+8} - s_b = a_{b+1} + a_{b+2} + \cdots + a_{b+8} \leq 16$$

we know

$$s_{b+8} \leq 16 + s_b < m + 16 + s_k$$

so s_{b+1} through s_{b+8} are all elements of U.

Now we know that T contains at least 9 partial sums from our sequence and U contains at least 8 partial sums. By the Pigeonhole Principle, one of the sixteen sets

$$\{s_k, m + s_k\}, \{1 + s_k, m + 1 + s_k\}, \ldots \{15 + s_k, m + 15 + s_k\}$$

contains two partial sums. These two give us the desired s_i, s_j with difference m, so the proof is complete.

5/3/25. For any positive integer $b \geq 2$, we write the base-b numbers as follows:

$$(d_k d_{k-1} \dots d_0)_b = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_1 b^1 + d_0 b^0,$$

where each digit d_i is a member of the set $S = \{0, 1, 2, \dots, b-1\}$ and either $d_k \neq 0$ or $k = 0$. There is a unique way to write any nonnegative integer in the above form.

If we select the digits from a different set S instead, we may obtain new representations of all positive integers or, in some cases, all integers. For example, if $b = 3$ and the digits are selected from $S = \{-1, 0, 1\}$, we obtain a way to uniquely represent all integers, known as the *balanced ternary* representation. As further examples, the balanced ternary representation of the numbers 5, -3, and 25 are:

$$5 = (1\ {-1}\ {-1})_3, \qquad -3 = (-1\ 0)_3, \qquad 25 = (1\ 0\ {-1}\ 1)_3.$$

However, not all digit sets can represent all integers. If $b = 3$ and $S = \{-2, 0, 2\}$, then no odd number can be represented. Also, if $b = 3$ and $S = \{0, 1, 2\}$ as in the usual base-3 representation, then no negative number can be represented.

Given a set S of four integers, one of which is 0, call S a *4-basis* if every integer n has at least one representation in the form

$$n = (d_k d_{k-1} \dots d_0)_4 = d_k 4^k + d_{k-1} 4^{k-1} + \cdots + d_1 4^1 + d_0 4^0,$$

where $d_k, d_{k-1}, \dots, d_0$ are all elements of S and either $d_k \neq 0$ or $k = 0$.

(a) Show that there are infinitely many integers a such that $\{-1, 0, 1, 4a+2\}$ is not a 4-basis.

(b) Show that there are infinitely many integers a such that $\{-1, 0, 1, 4a+2\}$ is a 4-basis.

Solution

Given a set S of 4 integers with one of them 0, we will say a nonzero integer n is **representable using** S if we can write n in the form

$$(d_k d_{k-1} \dots d_0)_4 = d_k 4^k + d_{k-1} 4^{k-1} + \cdots + d_1 4^1 + d_0 4^0,$$

where $d_k, d_{k-1}, \dots, d_0$ are all elements of S and $d_k \neq 0$.

Part a:

We will show that if $4a + 2$ is a multiple of 3, we do not get a 4-basis. Any value of $a \equiv 1 \pmod 3$ will force $3 \mid 4a+2$ so there are infinitely many such S. Let $b = \dfrac{4a+2}{3}$ and suppose that $S = \{-1, 0, 1, 3b\}$ is a 4-basis. Since $-b$ is representable using X,

$$-b = (d_k d_{k-1} \dots d_0)_4 = d_k 4^k + d_{k-1} 4^{k-1} + \cdots + d_1 4^1 + d_0 4^0,$$

where $d_k, d_{k-1}, \ldots, d_0$ are all elements of X and $d_k \neq 0$. Choose some representation with k minimal. Notice that

$$-b \equiv 3b \equiv 4a + 2 \equiv 2 \pmod 4$$

but our expression for b also gives

$$-b \equiv (d_k d_{k-1} \ldots d_0)_4 \equiv d_0 \pmod 4.$$

The only choice for d_0 in S that is congruent to 2 modulo 4 is $d_0 = 3b$. Then removing the final digit we get

$$-b = (d_k d_{k-1} \ldots d_0)_4 = 4 \cdot (d_k d_{k-1} \ldots d_1)_4 + 3b.$$

Isolating $-b$ gives $-b = (d_k d_{k-1} \ldots d_1)_4$. However, this gives us a representation for $-b$ with fewer digits, contradicting the minimality of k. So there could not have been a representation of $-b$, and $\{-1, 0, 1, 4a + 2\}$ is not a 4-basis.

Part b:

We claim that $S_b = \{-1, 0, 1, 4^b - 2\}$ is a 4-basis for every positive integer b. Since $4^b - 2$ is of the form $4a + 2$, this will suffice.

For a given n, let d' be the element of S_b such that $n \equiv d' \pmod 4$. Note that if

$$\frac{n - d'}{4} = d_k 4^k + d_{k-1} 4^{k-1} + \cdots + d_1 4^1 + d_0$$

then

$$n = d_k 4^{k+1} + d_{k-1} 4^k + \cdots + d_1 4^2 + d_0 4^1 + d'.$$

Therefore,

Lemma 1: If $\frac{n-d'}{4}$ is representable by S_b, then n is representable by S_b.

The number $|\frac{n-d'}{4}|$ will often be smaller than $|n|$, suggesting that we may be able to use the fact that n is representable when $|n|$ is small to prove that every n is representable. We first explore when $|\frac{n-d'}{4}| < |n|$.

Assume that $n \neq 0$. If $d' = -1$, 0, or 1, then

$$\left|\frac{n - d'}{4}\right| \leq \frac{|n| + 1}{4} \leq |n|.$$

However, if $d' = 4^c - 2$ then

$$\left|\frac{n - d'}{4}\right| \leq \frac{|n| + (4^c - 2)}{4}$$

which could be larger than n. In the case that $|n| \geq \frac{4^c}{3}$, we have $4^c \leq 3|n|$ so

$$\left|\frac{n - d'}{4}\right| \leq \frac{|n| + (4^c - 2)}{4} < \frac{|n| + 4^c}{4} \leq \frac{|n| + 3|n|}{4} \leq |n|.$$

In summary, we see that:

Lemma 2: If $|n| \geq \frac{4^b}{3}$ then $|\frac{n-d'}{4}| < |n|$.

Suppose that we know that every n with $0 < |n| < \frac{4^b}{3}$ is representable by S_b. Then we can prove that every n is representable by strong induction on $|n|$ as follows. Suppose that n is representable when $|n| \leq N$, and suppose that $|n| = N+1$. If $\frac{n-d'}{4} = 0$ then n is in S_b so is representable. Assume not. By the first lemma, n is representable if $\frac{n-d'}{4}$ is representable. Either $0 < |\frac{n-d'}{4}| < \frac{4^b}{3}$, in which case $\frac{n-d'}{4}$ is representable by assumption, or $|\frac{n-d'}{4}| \geq \frac{4^b}{3}$, in which case $|\frac{n-d'}{4}| < |n|$, by our second lemma, so that $\frac{n-d'}{4}$ is representable by our inductive hypothesis. So to prove that every n is representable, it suffices to prove that every n with $0 < |n| < \frac{4^b}{3}$ is representable.

As an example, the $b = 1$ case is follows since values of n for which $0 < |n| < \frac{4}{3}$ are -1 and 1, and these values are in S_1.

For values of n such that $-n$ is representable by S_1 with $k \leq b$ digits, we can use such a representation to construct a representation of n by S_b as follows. Suppose that

$$-n = d_{k-1}4^{k-1} + d_{k-2}4^{k-2} + \cdots + d_1 4^1 + d_0 4^0$$

where $d_0, d_1, \ldots, d_{k-1} \in S_1$ and $k \leq b$. Then

$$n = (-d_{k-1})4^{k-1} + (-d_{k-2})4^{k-2} + \cdots + (-d_1)4^1 + (-d_0)4^0$$

We want to use this to represent n by S_b. Consider a new sum,

$$e_{k+b-1}4^{k+b-1} + e_{k+b-2}4^{k+b-2} + \cdots + e_0 4^0$$

where e_i is defined as follows. If $i < k$ then

$$e_i = \begin{cases} -d_i & \text{if } d_i \in \{-1, 0, 1\} \\ 4^b - 2 & \text{if } d_i = 2 \end{cases}$$

If $k \leq i < b$ then we set $e_i = 0$. When $i \geq b$, let

$$e_i = \begin{cases} 0 & \text{if } d_{i-b} \in \{-1, 0, 1\} \\ -1 & \text{if } d_{i-b} = 2 \end{cases}$$

Then if $d_i \neq 2$,

$$e_{i+b}4^{i+b} + e_i 4^i = 0 + (-d_i)4^i = (-d_i)4^i.$$

Furthermore, if $d_i = 2$, we get

$$e_{i+b}4^{i+b} + e_i 4^i = (-1)4^{i+b} + (4^b - 2)4^i = -2 \cdot 4^i = (-d_i)4^i$$

as well. Therefore

$$\begin{aligned} n &= \sum_{i=0}^{k-1}(-d_i)4^i \\ &= \sum_{i=0}^{k-1} e_{i+b}4^{i+b} + e_i 4^i \\ &= \sum_{i=0}^{k+b-1} e_i 4^i. \end{aligned}$$

Each e_i is in S_b so we are done.

We now use mathematical induction to prove that every n with $|n| < \frac{4^b}{3}$ can be represented by S_1 using b or fewer digits.

The $b = 1$ case is that ± 1 may be represented with a single digit. This is true since these are indeed in S_1. Now we suppose that the claim is true for b and prove it is true for $b+1$. That is, if $\frac{n-d'}{4}$ is representable using b digits, then n is representable using $b+1$ digits. So to prove the claim, it suffices to show that if $|n| < \frac{4^{b+1}}{3}$ then $|\frac{n-d'}{4}| < \frac{4^b}{3}$.

If $d' \in \{-1, 0, 1\}$ then $n - d'$ is the nearest multiple of 4 to n. If $d' = 2$ then $n - d'$ is the next lower multiple of 4. So replacing n by $n - d'$ rounds n to the nearest multiple of 4, rounding down to break ties.

Suppose that $|n| < \frac{4^{b+1}}{3}$. Since n is an integer and 4^{b+1} is one more than a multiple of 3,

$$\frac{1 - 4^{b+1}}{3} \leq n \leq \frac{4^{b+1} - 1}{3}.$$

When we round all elements in this interval to the nearest multiple of 4 we find

$$\frac{4 - 4^{b+1}}{3} \leq n - d' \leq \frac{4^{b+1} - 4}{3}.$$

This means

$$\frac{1 - 4^b}{3} \leq \frac{n - d'}{4} \leq \frac{4^b - 1}{3}.$$

making $\left|\frac{n - d'}{4}\right| < \frac{4^b}{3}$. So if $|n| < \frac{4^{b+1}}{3}$, then $|\frac{n-d'}{4}| < \frac{4^b}{3}$. This proves that every n with $|n| < \frac{4^b}{3}$ is representable by S_1 using no more than b digits.

Since every n with $|n| < \frac{4^b}{3}$ is representable by S_1 using at most b digits, every such n is representable in S_b and this gives us the base cases for our induction.