1/1/25. Alex is trying to open a lock whose code is a sequence that is three letters long, with each of the letters being one of A, B or C, possibly repeated. The lock has three buttons, labeled A, B and C. When the most recent 3 button-presses form the string, the lock opens. What is the minimum number of total button presses Alex needs to be sure to open the lock?

Solution

The answer is 29. This can be achieved with the following sequence of presses:

$$AAACCCBCCACBBCBACABCAABBBABAA$$

There are $3 \cdot 3 \cdot 3 = 27$ different strings of three letters with each letter being one of A, B, or C. All 27 of these strings appear consecutively in the above sequences of presses.

Each press of the button corresponds to attempting at most one more string, namely the one formed by the previous three presses. No string can be attempted after the first two presses. Therefore, the first time that all 27 strings can be tried on the lock is after $27 + 2 = 29$ presses of the button, so 29 is indeed the minimum.

2/1/25. In the 5×6 grid shown, fill in all of the grid cells with the digits 0–9 so that the following conditions are satisfied:

1. Each digit gets used exactly 3 times.
2. No digit is greater than the digit directly above it.
3. In any four cells that form a 2×2 subgrid, the sum of the four digits must be a multiple of 3.

				7	
	8				6
		2	4		
5				1	
	3				

You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

Label the unknown cells with the variables a through v as shown to the right. Using rule 3 repeatedly in the top two rows, we have

a	b	c	d	7	e
f	8	g	h	i	6
j	k	2	4	l	m
5	n	o	p	1	q
r	3	s	t	u	v

$$b+8+c+g \equiv c+g+d+h \equiv d+h+7+i \equiv 7+i+e+6 \equiv 0 \pmod 3.$$

The first equivalence above gives us $b+8 \equiv d+h \pmod 3$, and the third equivalence gives us $d+h \equiv e+6 \pmod 3$. Thus $b+8 \equiv e+6 \pmod 3$. Since $8 \le b \le 9$ and $6 \le e \le 9$, we must have that (b, e) is either $(8, 7)$ or $(9, 8)$.

Applying the technique above repeatedly gives us the following fourth rule:

4. If two columns are an even distance apart, then any adjacent pair of cells in the first column has the same sum modulo 3 as the corresponding adjacent pair in the second column. Similarly, if two rows are an even distance apart, then any adjacent pair of cells in the first row has the same sum modulo 3 as the corresponding adjacent pair in the second row.

Using rule 4, we have $5 + r \equiv 1 + u \pmod 3$. In particular, since u must be 0 or 1, we must have r be either 0 or 2 modulo 3.

Now consider the sum of all numbers of the board. We have three copies of every digit, so this sum is a multiple of 3. We also know that the six 2×2 squares shown in the diagram to the right all sum to a multiple of 3. Subtracting these six squares out from the sum of the whole board, we get that

a	b	c	d	7	e
f	8	g	h	i	6
j	k	2	4	l	m
5	n	o	p	1	q
r	3	s	t	u	v

$$7 + e + 2 + 4 + r + 3 = 16 + e + r \equiv 1 + e + r \equiv 0 \pmod 3.$$

But e must be 7 or 8 from above, and r must be 0 or 2 modulo 3, so the only possibility is that $e = 8$ and $r \equiv 0 \pmod 3$. Hence $(b, e) = (9, 8)$ and also $u = 1$.

Now the grid looks as shown. Next, note that $i \equiv 0 \pmod 3$ using the upper-right 2×2 corner of the grid. So by rule 4 applied to the top two rows, we have

a	9	c	d	7	8
f	8	g	h	i	6
j	k	2	4	l	m
5	n	o	p	1	q
r 0 or 3	3	s	t	1	v

$$\begin{aligned} a + f \equiv c + g \equiv 7 + i \equiv 1 \pmod 3, \\ 9 + 8 \equiv d + h \equiv 8 + 6 \equiv 2 \pmod 3. \end{aligned}$$

In particular, no column can contain two 9's, so all three 9's must be in the top row. Additionally, by rule 4, $c + d \equiv 2 + 4 \equiv 0 \pmod 0$, so since one of c and d must be a 9, they both must be multiples of 3. Also a is a multiple of 3 since by rule 4, $a + 9 \equiv r + 3 \equiv 0 \pmod 3$. So a, c, d, i, and r are all multiples of 3.

At this point, we have shaded the boxes that we know must be multiples of 3. We know enough of the cells modulo 3 for simple applications of rules 3 and 4 to give us the value of all of the cells modulo 3. For example, $d + 7 + h + i \equiv 0 + 1 + h + 0 \equiv 0 \pmod 3$, so $h \equiv 2 \pmod 3$. Repeating this around the grid gives us the following chart:

a	9	c	d	7	8
f	8	g	h	i	6
j	k	2	4	l	m
5	n	o	p	1	q
r	3	s	t	1	v

0 mod 3	9	0 mod 3	0 mod 3	7	8
1 mod 3	8	1 mod 3	2 mod 3	0 mod 3	6
2 mod 3	1 mod 3	2	4	0 mod 3	0 mod 3
5	1 mod 3	2 mod 3	1 mod 3	1	2 mod 3
0 mod 3	3	0 mod 3	0 mod 3	1	2 mod 3

We now just need to apply rules 1 and 2 repeatedly to determine the exact values of each cell. This is mostly a case of filling in from the top and/or the bottom using rule 2, and keeping track of how many times each digit is used to apply rule 1. To start, the only place for the three 0's is the bottom row, and the only place for the remaining 1 is the 4th column.

The grid now looks as at right. The only numbers that fit in the left column with the correct residues modulo 3 are $(a, f, j) = (9, 7, 5)$, and the only remaining place for the last 8 is h. This makes $d = 9$ and $c = 6$. Then $o = 2$ and $(q, v) = (5, 2)$ are the only ways to place the remaining numbers that are 2 modulo 3, and the rest of the grid easily fills to give us the final answer below.

a	9	c	d	7	8
f	8	g	h	i	6
j	k	2	4	l	m
5	n	o	1	1	q
0	3	0	0	1	v

9	9	6	9	7	8
7	8	4	8	3	6
5	7	2	4	3	6
5	4	2	1	1	5
0	3	0	0	1	2

3/1/25. An infinite sequence of positive real numbers $a_1, a_2, a_3, \ldots$ is called *territorial* if for all distinct positive integers i, j with $i < j$, we have $|a_i - a_j| \geq \frac{1}{j}$. Can we find a territorial sequence $a_1, a_2, a_3, \ldots$ for which there exists a real number c with $a_i < c$ for all i?

Solution

We will construct a territorial sequence exists whose maximum value is 2. This allows us to choose c to be any number greater than 2.

Consider the sequence

$$(a_i) = 2, 1, \frac{1}{2}, \frac{3}{2}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{9}{8}, \frac{11}{8}, \frac{13}{8}, \frac{15}{8}, \frac{1}{16}, \frac{3}{16}, \ldots.$$

Specifically, $a_1 = 2$, and for any positive integer $n \geq 2$, let k be the unique nonnegative integer such that $2^k < n \leq 2^{k+1}$, and then $a_n = \frac{2(n-2^k)-1}{2^k}$. Note that all the numbers in the sequence are distinct.

Suppose we are given two positive integers i, j with $i < j$. Let k be the unique nonnegative integer such that $2^k < j \leq 2^{k+1}$. Then $a_j = \frac{t}{2^k}$ for some odd integer t. Since $i \leq 2^{k+1}$, we also have $a_i = \frac{u}{2^k}$ for some integer u with $t \neq u$ and u not necessarily odd. Then $|a_i - a_j| \geq \frac{1}{2^k} > \frac{1}{j}$, as desired.

4/1/25. Bunbury the bunny is hopping on the positive integers. First, he is told a positive integer n. Then Bunbury chooses positive integers a, d and hops on all of the spaces $a, a+d, a+2d, \dots, a+2013d$. However, Bunbury must make these choices so that the number of every space that he lands on is less than n and relatively prime to n.

A positive integer n is called *bunny-unfriendly* if, when given that n, Bunbury is unable to find positive integers a, d that allow him to perform the hops he wants. Find the maximum bunny-unfriendly integer, or prove that no such maximum exists.

Solution

Let M be the product of all prime numbers less than 2014. We claim that the maximum bunny-unfriendly integer is $2013M$. Note that $2013 = 3 \cdot 11 \cdot 63$ is not prime, so all of the prime divisors of $2013M$ are less than 2013.

First, we verify that $2013M$ is bunny-unfriendly. Suppose, for sake of contradiction, that positive integers a and d could be chosen so that $a+2013d < 2013M$ and $\gcd(a+kd, 2013M) = 1$ for all $0 \le k \le 2013$. If there is a prime p less than 2014 that does not divide d, then d^{-1} exists modulo p. Choose $0 \le k < p$ such that $k \equiv -ad^{-1} \pmod{p}$. Then $a+kd$ is a multiple of p, and $\gcd(a+kd, 2013M) \ge p$, a contradiction. Therefore no such p exists, and thus d must be a multiple of M. But then $a + 2013d > 2013d > 2013M$, also a contradiction. Thus, Bunbury will not be able to find an a and d to use for his hopping, and hence $2013M$ is bunny-unfriendly.

Next, we verify that all numbers greater than $2013M$ are bunny-friendly. Let n be an integer greater than $2013M$, and let p be the largest prime divisor of n. We break into cases based on the properties of p.

Case 1: $p < 2014$, so that all the prime divisors of n are less than 2014.
Let x be the product of n's distinct prime divisors. Let $a = 1$ and $d = x$, and note that we have that $\gcd(n, 1+kx) = 1$ for all integers k. Furthermore, $1 + 2013x < 2014x$, which is at most n because $\frac{n}{x} > \frac{n}{M} > 2013$. Thus, this choice of a and d proves that n is bunny-friendly.

Case 2: $p > 2014$.
We first consider $a = 1$ and $d = \frac{n}{p}$. If d is still a multiple of p, then $\gcd(n, 1+kd) = 1$ for all integers k. Furthermore, $1 + 2013d = 1 + \frac{2013n}{p} < n$. So this choice of a and d proves that n is bunny-friendly.

But if d is not a multiple of p, then there is a unique b with $0 \le b < p$ such that $1 + bd$ is a multiple of p, and hence $\gcd(n, 1+bd) = p$. For all other $0 \le k < p$ with $k \ne b$, we have $\gcd(n, 1+kd) = 1$. Thus, if $b > 2013$, choosing $a = 1$ and $d = \frac{n}{p}$ still proves that n is bunny-friendly, by the argument above.

If $b < 2014$, then we can try to start our hopping after the "bad" multiple of d. That is, we let $a = 1 + (b+1)d$. This works provided that

$$a + 2013d = 1 + (b + 2014)d < n.$$

So it is sufficient to have $1 + 4027d < n$. If $p > 4028$, then this will always be the case, because then $1 + 4027d < 1 + \frac{4027}{4028}n < n$.

So our remaining case is when d is not a multiple of p and $p \leq 4027$. In this case we have to try something different. Let $a = p+d$. Note that $\gcd(n, p+(k+1)d) = 1$ for all $0 \leq k \leq 2013$. Furthermore,

$$p + 2014d = p + \frac{2014n}{p} = \left(\frac{p}{n} + \frac{2014}{p}\right)n,$$

but $\frac{p}{n}$ is very small (recall that $n > 2013M$ is very large relative to $p < 4028$); in particular $n > p^2$, so $\frac{p}{n} < \frac{1}{p}$, and hence

$$p + 2014d < \left(\frac{1}{p} + \frac{2014}{p}\right)n = \frac{2015}{p}n < n.$$

Thus this choice of a and d proves that n is bunny-friendly.

In all cases, any $n > 2013M$ is bunny-friendly. Thus, $2013M$ is the largest bunny-unfriendly integer.

USA Mathematical Talent Search

Round 1 Solutions

Year 25 — Academic Year 2013–2014

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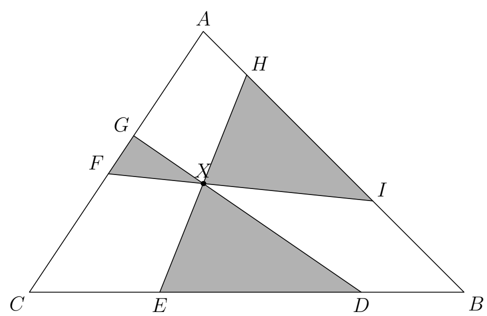
5/1/25. Niki and Kyle play a triangle game. Niki first draws $\triangle ABC$ with area 1, and Kyle picks a point X inside $\triangle ABC$. Niki then draws segments $\overline{DG}$, $\overline{EH}$, and $\overline{FI}$, all through X, such that D and E are on $\overline{BC}$, F and G are on $\overline{AC}$, and H and I are on $\overline{AB}$. The ten points must all be distinct. Finally, let S be the sum of the areas of triangles DEX, FGX, and HIX. Kyle earns S points, and Niki earns $1 - S$ points. If both players play optimally to maximize the amount of points they get, who will win and by how much?

Solution

(We use the common notation that $[PQR]$ denotes the area of triangle PQR.)

We claim that the game will end with $S = \frac{1}{3}$. In particular, for any X that Kyle chooses, we will show that Niki can choose her points so that $S \le \frac{1}{3}$, and if X is the centroid of ABC, then the best that Niki can do is $S = \frac{1}{3}$.

Note that none of the argument below depends on how triangle ABC is drawn, so how Niki draws ABC is irrelevant.



In the diagram at right, S is the sum of the shaded areas. Consider the sum of the areas $[AFI]+[BEH]+[CDG]$. Notice that this sum counts each white region once but each shaded region twice: for example, $[HIX]$ is counted in both $[AFI]$ and $[BEH]$. Therefore,

$$[AFI] + [BEH] + [CDG] = 1 + S,$$

and hence

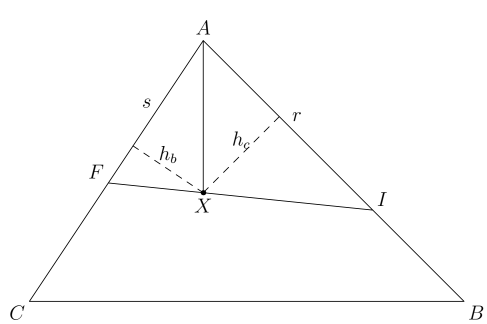
$$S = [AFI] + [BEH] + [CDG] - 1.$$

We will show that for any X Kyle chooses, Niki can choose her points such that

$$[AFI] + [BEH] + [CDG] \le \frac{4}{3},$$

which implies that $S \le \frac{1}{3}$.

Let X be Kyle's chosen point. Let a, b, and c be the lengths of $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively, and let h_A, h_B, and h_C be the distances from X to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively.



We will focus on Niki's choices of F and I to minimize $[AFI]$. Let $r = AI$ and $s = AF$. On the one hand,

$$[AFI] = [AFX] + [AIX] = \frac{1}{2}(rh_B + sh_C).$$

On the other hand, since $[ACB] = 1$, we have

$$[AFI] = \frac{[AFI]}{[ACB]} = \frac{rs \sin \angle FAI}{bc \sin \angle CAB} = \frac{rs}{bc}.$$

Combining the above two equations, we have

$$[AFI] = \frac{1}{2}(rh_B + sh_C) = \frac{rs}{bc}. \qquad (*)$$

Set $x = [AFI]$. By the AM-GM inequality,

$$x = \frac{1}{2}(rh_B + sh_C) \geq \sqrt{rh_b sh_C}.$$

But $rs = \frac{1}{2}bc(rh_B + sh_C) = bcx$ by $(*)$, therefore

$$x \geq \sqrt{bch_B h_C x}.$$

Squaring and solving for x gives

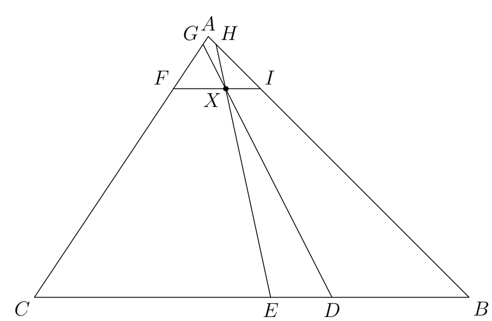
$$x \geq bch_B h_C,$$

with equality if and only if $rh_B = sh_C$. In the equality case, $(*)$ gives $rh_B = \frac{rs}{bc}$, so $s = bch_B$, and similarly $r = bch_C$. Thus, Niki minimizes $[AFI]$ by choosing F and I so that $r = bch_C$ and $s = bch_B$. She will able to make these choices provided that $bh_B \leq 1$ and $ch_C \leq 1$.

So, if Kyle picks X such that ah_A, bh_B, and ch_C are all less than 1, then Niki can choose her points so that

$$[AFI] + [BEH] + [CDG] = bch_B h_C + cah_C h_A + abh_A h_B.$$

Let's dispose of the contrary case first: suppose Kyle picks X such that (without loss of generality) $ah_A \geq 1$, or equivalently $h_a \geq \frac{1}{a}$. Niki can choose F and I so that $\overline{FI} \parallel \overline{CB}$. Then by similarity, and using the fact that the height from A to $\overline{BC}$ is $\frac{2}{a}$, we have



$$[AFI] = \left(\frac{\frac{2}{a} - h_a}{\frac{2}{a}}\right)^2 \leq \left(\frac{\frac{1}{a}}{\frac{2}{a}}\right)^2 = \frac{1}{4}.$$

Furthermore, Niki can pick G and H very close to A so that $[BEH] + [CDG]$ is as close to 1 as she wishes (in the extreme case, $G = H = A$ and $[BEH] + [CDG] = 1$), in particular, Niki can get $[BEH] + [CDG] < \frac{13}{12}$. Thus, Niki can achieve

$$[AFI] + [BEH] + [CDG] < \frac{1}{4} + \frac{13}{12} = \frac{4}{3}.$$

Now for the remaining cases: let $p = ah_A$, $q = bh_B$, and $r = ch_C$, and assume that Kyle picks X such that p, q, r are all less than 1. Then we have

$$[AFI] + [BEH] + [CDG] = qr + rp + pq.$$

But we also have the general inequality

$$(qr + rp + pq) \leq \frac{1}{3}(p + q + r)^2,$$

with equality if and only if $p = q = r$. (This is a corollary of the rearrangement inequality $pq+qr+rp \leq p^2+q^2+r^2$, which itself follows from the inequality $(p-q)^2+(q-r)^2+(r-p)^2 \geq 0$.) On the other hand,

$$p + q + r = 2[BCX] + 2[CAX] + 2[ABX] = 2[ABC] = 2,$$

so Niki's choices result in

$$[AFI] + [BEH] + [CDG] = qr + rp + pq \leq \frac{1}{3}(p + q + r)^2 = \frac{4}{3},$$

and this inequality is strict unless $p = q = r$. Since Kyle wants to choose X to maximize the amount of area that Niki must choose, he wants to force the inequality to be an inequality by choosing X such that $p = q = r$. This means that $ah_A = bh_B = ch_C$, which means that X is the centroid of ABC.

In all cases, Niki can pick points such that

$$S = [AFI] + [BEH] + [CDG] - 1 \leq \frac{1}{3},$$

and Kyle can force her into $S = \frac{1}{3}$ by picking X to be the centroid of ABC. Thus, with optimal play, Niki will score $\frac{2}{3}$ and Kyle will score $\frac{1}{3}$.