USA Mathematical Talent Search

Round 3 Solutions

Year 24 — Academic Year 2012–2013

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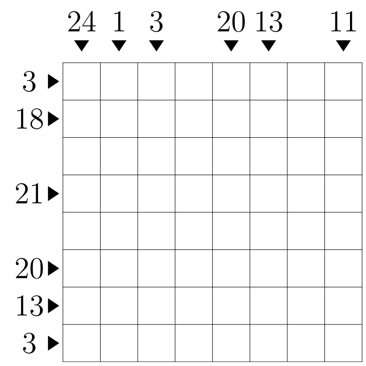
1/3/24. In the 8×8 grid shown, fill in 12 of the grid cells with the numbers 1–12 so that the following conditions are satisfied:



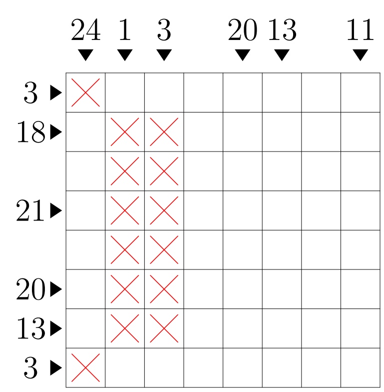
1. Each cell contains at most one number, and each number from 1–12 is used exactly once.
2. Two cells that both contain numbers may not touch, even at a point.
3. A clue outside the grid pointing at a row or column gives the sum of all of the numbers in that row or column. Rows and columns without clues have an unknown sum.

You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

Number the rows and columns 1–8 starting from the top and left, and denote the space in row m and column n by RmCn. For example, the top left square is R1C1. This solution will solve the problem without using the 18 clue in row 2, which is actually unnecessary to have a unique solution.

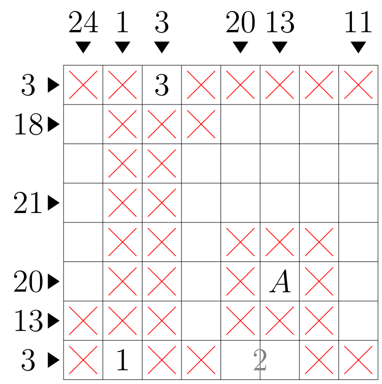
The only ways to have a row or column with numbers summing to 3 is for it to have only 3 or only 1 and 2. The only way to have a row or column with numbers summing to 1 is to have just 1. From this, we conclude that column 2 has just 1, and that column 3 has just 3, since it cannot have 1.



Now consider the first and last row. One of them has only 3, and the other one has only 1 or 2. The one with 3 must have it in column 3. The one with 1 and 2 has the 1 in column 2. As a result, the 1 in column 2 and the 3 in column 3 must be in the first and last rows. The other spaces of these two columns cannot have numbers. To the right is an image summarizing the progress so far.

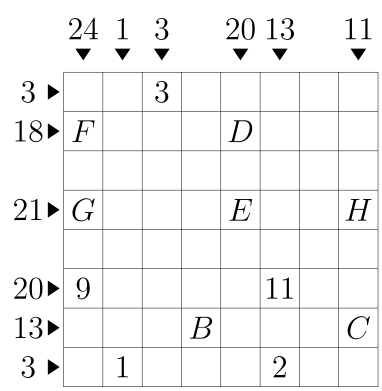
Notice that by the second rule, a 2×2 subblock of the grid can contain at most one number. Consider rows 6 and 7, which have 20 and 13 as clues. Since we are not allowed to have numbers 13 or greater, each row must have at least two numbers. But since the two rows can be partitioned into four 2×2 subblocks each with at most number, we can have at most four numbers in the two rows, so each row has exactly two numbers. The same logic can be applied to columns 5 and 6 with 20 and 13 as clues to show both columns have exactly two numbers.

Suppose now that R6C5, in the intersection of the row and column with 20, had a number a. Then the second number in row 6 must be $20 - a$, and the same is true of column 5. Since we can't have the same number twice, this is a contradiction, so R6C5 cannot have a number. Similarly, R7C6 cannot have a number either.



Looking at rows 6 and 7 again, and accounting for the fact that columns 2 and 3 can't have numbers in these two rows, we notice that column 1 can have at most one number, and the only way to fit 3 numbers in the last five columns without any touching is to put a number in each of column 4, column 6, and column 8. Above, we determined R7C6 cannot have a number, so the only place for a number in column 6 is R6C6; call this number A. Now consider how we might fit four numbers in columns 5 and 6. The first four rows can have at most two numbers. Row 6 has a number, so the only place for the fourth number is row 8. Above, we determined that the only number in rows 1 and 8 that can occur outside of columns 2 and 3 is a 2. Therefore, a 2 appears in either R8C5 or R8C6. This means row 8 has the 1 and 2 and row 1 has the 3. The shown image summarizes our deductions so far.

We know row 7 has two numbers, but there are only two spaces that remain. Therefore these two spaces, R7C4 and R7C8, have numbers. This means the two cells in row 6 with numbers are R6C1 and R6C6. Also, to avoid touching R7C4, the 2 must go in R8C6. Since column 6 must sum to 13, its second number must be 11, which is the value of A in R6C6. Then in row 6, with sum 20, the second number in R6C1 must be a 9. Now notice that the only way to fit two numbers into column 5 is to put them both in R2C5 and R4C5. Furthermore, since two distinct numbers from 1 to 12 can sum to at most 23, column 1 must have at least three numbers, and the only way to fit them is to use rows 2, 4, 6.



At this point, we have determined the positions of eleven numbers, and there is only one more. Currently column 8 has only one number, but it needs two since the 11 has been placed already. Similarly, row 4 has only two numbers, and the only two number combinations summing to 21 are $9+12$ and $10+11$. Since the 11 and 9 have been used, neither works, so it needs a third number. Combining these, we see the 12th number must be in R4C8. The image shows the grid with all numbers placed and gives names to all of the numbers which we have yet to determine the value of.

Notice that B is the only number not clued by any of the column sums. The column sums total to 72, which is 6 less than 78, the sum of the numbers 1 through 12. Therefore $B = 6$. Using the 13 clue in row 7 and the 11 clue in column 8, we then get $C = 7$ and $H = 4$.

The four numbers that remain are $5, 8, 10, 12$. We have $D + E = 20$, $E + G = 21 - 4 = 17$, and $F + G = 24 - 9 = 15$. By considering the possible sums of pairs of these four numbers, all of which are distinct, we find that D, E can only be $8, 12$ in some order, E, G are $5, 12$ in some order, and F, G are $5, 10$ in some order. Together, this information gives $D = 8$, $E = 12$, $G = 5$, and $F = 10$. This reaches the solved grid shown below, which is the only answer to the problem.

	24 ▼	1 ▼	3 ▼		20 ▼	13 ▼		11 ▼
3 ▶			3					
18 ▶	10				8			
21 ▶	5				12			4
20 ▶	9					11		
13 ▶				6				7
3 ▶		1				2		

Remark: As noted, this solution did not make use of the 18 clue in row 2. If one takes it into account, then the row clues sum to 78, so there can be no numbers in rows 3 and 5. This provides some alternate, easier methods to find the positions of the twelve numbers in the grid.

2/3/24. Palmer and James work at a dice factory, placing dots on dice. Palmer builds his dice correctly, placing the dots so that 1, 2, 3, 4, 5, and 6 dots are on separate faces. In a fit of mischief, James places his 21 dots on a die in a peculiar order, putting some nonnegative integer number of dots on each face, but not necessarily in the correct configuration. Regardless of the configuration of dots, both dice are unweighted and have equal probability of showing each face after being rolled.

Then Palmer and James play a game. Palmer rolls one of his normal dice and James rolls his peculiar die. If they tie, they roll again. Otherwise the person with the larger roll is the winner. What is the maximum probability that James wins? Give one example of a peculiar die that attains this maximum probability.

Solution

Let p be the probability James wins the game after it ends. Note that if Palmer and James tie on a roll, then by definition James has probability p of winning in the subsequent rolls that follow. Let $f(k)$ be the probability that if James rolls a k on his die, he will eventually be the winner. Note that $f(0) = 0$ and $f(k) = 1$ for $k > 6$. For $k \in \{1, 2, 3, 4, 5, 6\}$, we have $f(k) = \frac{(k-1)+p}{6}$. This is because Palmer's die has $k-1$ numbers less than k, and there is also a $\frac{1}{6}$ chance of tying. Then

$$p = \frac{f(a_1) + f(a_2) + \cdots + f(a_6)}{6}.$$

Suppose James has exactly m faces with a positive number of dots on his die. Notice that if $m \le 3$, then at least half of the faces on James's die lose automatically, and his probability of winning is at most $\frac{1}{2}$. For $m \in \{4, 5, 6\}$, assume $a_1, a_2, \ldots, a_m$ are the nonzero faces of the die and note that $f(k) \le \frac{(k-1)+p}{6}$ for all $k > 0$. Then we have

$$p = \frac{f(a_1) + f(a_2) + \cdots + f(a_m)}{6} \le \sum_{i=1}^{m} \frac{(a_i - 1) + p}{36} = \frac{21 - m + mp}{36}.$$

This simplifies to $p \le \frac{21-m}{36-m}$. For $m \in \{4, 5, 6\}$, the maximum value this expression takes is $\frac{17}{32}$. As this is higher than $\frac{1}{2}$, we have that $p \le \frac{17}{32}$ in all cases.

We now construct a die obtaining this probability. Note that our bound $f(k) \le \frac{(k-1)+p}{6}$ has equality if $k \in \{1, 2, 3, 4, 5, 6\}$ and is a strict inequality if $k > 6$. Let $a_5 = a_6 = 0$ and a_1, a_2, a_3, a_4 be numbers in $\{1, 2, 3, 4, 5, 6\}$ summing to 21, such as $6, 6, 6, 3$. Then $p = \frac{17+4p}{36}$, which simplifies to $p = \frac{17}{32}$. Therefore the upper bound we found is achievable, and this is the maximum probability.

Remark: Using the result of the last paragraph, we can verify that a die obtains the maximum probability $\frac{17}{32}$ only when $(a_1, a_2, a_3, a_4, a_5, a_6)$ is a permutation of one of

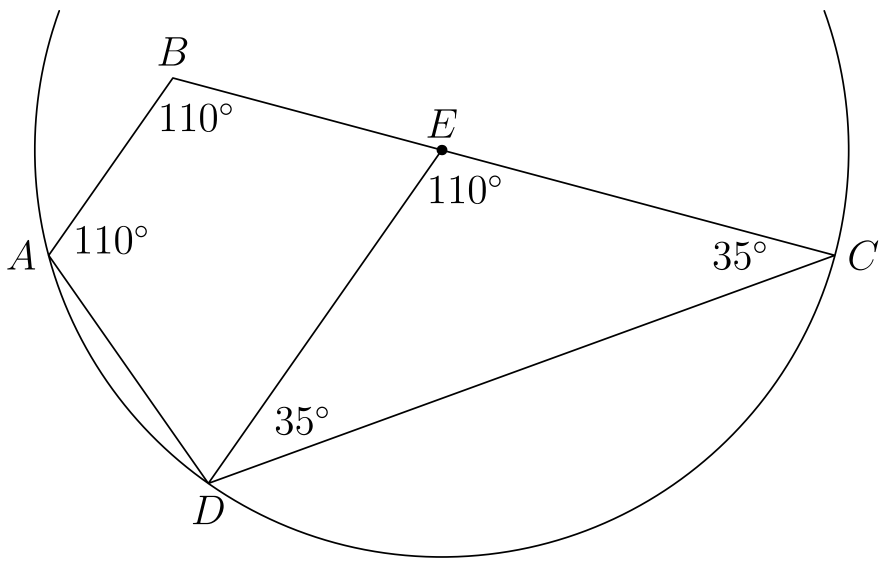
$$(6, 6, 6, 3, 0, 0), \qquad (6, 6, 5, 4, 0, 0), \qquad (6, 5, 5, 5, 0, 0).$$

3/3/24. In quadrilateral $ABCD$, $\angle DAB = \angle ABC = 110^\circ$, $\angle BCD = 35^\circ$, $\angle CDA = 105^\circ$, and AC bisects $\angle DAB$. Find $\angle ABD$.

Solution

Take E on BC so that DE is parallel to AB. Since $\angle BAD = \angle ABE$ and AB is parallel to DE, quadrilateral $ABED$ is an isosceles trapezoid, which means it is cyclic, so $\angle ABD = \angle AED$.

Also, $\angle DEC = \angle ABE = 110^\circ$, and $\angle CDE = 180^\circ - \angle DEC - \angle ECD = 180^\circ - 110^\circ - 35^\circ = 35^\circ$, so triangle CDE is isosceles with $CE = DE$.



Since $\angle CAD = \frac{1}{2}\angle BAD = \frac{1}{2}\angle CED$, A lies on the circle centered at E with radius $CE = DE$, so $\angle AED = 2\angle ACD$. From triangle ACD, $\angle ACD = 180^\circ - \angle CAD - \angle ADC = 180^\circ - 55^\circ - 105^\circ = 20^\circ$, so $\angle AED = 2 \cdot 20^\circ = 40^\circ$, which means $\angle ABD = 40^\circ$.

4/3/24. Denote by $\lfloor x \rfloor$ the greatest integer less than or equal to x. Let $m \geq 2$ be an integer, and let s be a real number between 0 and 1. Define an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ by setting $a_1 = s$ and $a_k = ma_{k-1} - (m-1)\lfloor a_{k-1} \rfloor$ for all $k \geq 2$. For example, if $m = 3$ and $s = \frac{5}{8}$, then we get $a_1 = \frac{5}{8}$, $a_2 = \frac{15}{8}$, $a_3 = \frac{29}{8}$, $a_4 = \frac{39}{8}$, and so on.

Call the sequence $a_1, a_2, a_3, \ldots$ **orderly** if we can find rational numbers b, c such that $\lfloor a_n \rfloor = \lfloor bn + c \rfloor$ for all $n \geq 1$. With the example above where $m = 3$ and $s = \frac{5}{8}$, we get an orderly sequence since $\lfloor a_n \rfloor = \left\lfloor \frac{3n}{2} - \frac{3}{2} \right\rfloor$ for all n. Show that if s is an irrational number and $m \geq 2$ is any integer, then the sequence $a_1, a_2, a_3, \ldots$ is **not** an orderly sequence.

Solution

Let $\{z\}$ denote the fractional part of a real number z, and note that $\{z\} = z - \lfloor z \rfloor$. We rewrite the given recurrence as

$$a_k = \lfloor a_{k-1} \rfloor + m\{a_{k-1}\}.$$

Let the decimal expansion of $\{s\}$ in base m be $0.d_1d_2d_3\ldots$, written as $(0.d_1d_2d_3\ldots)_m$, where $0 \leq d_i \leq m-1$ for all i and the sequence is not eventually constant at $m-1$. First we show by induction that

$$\{a_k\} = (0.d_kd_{k+1}d_{k+2}\ldots)_m.$$

The base case $k = 1$ is true by how we've defined the d_k. Suppose it is true for k. Notice that if we take the fractional part of both sides of the recurrence, we get $\{a_{k+1}\} = \{ma_k\}$. Furthermore, $m \cdot (0.d_kd_{k+1}d_{k+2}\ldots)_m = (d_k.d_{k+1}d_{k+2}\ldots)_m$, which has fractional part $(0.d_{k+1}d_{k+2}d_{k+3}\ldots)_m$ as desired.

From the above claim, we obtain that $\lfloor m\{a_k\} \rfloor = d_k$. Taking the floor of both sides of the recurrence, we have

$$\lfloor a_k \rfloor = \lfloor a_{k-1} \rfloor + \lfloor m\{a_{k-1}\} \rfloor.$$

The last term is equal to d_{k-1}. Therefore,

$$\lfloor a_k \rfloor - \lfloor a_{k-1} \rfloor = d_{k-1}. \tag{1}$$

Suppose the sequence $a_1, a_2, a_3, \ldots$ is orderly. We will show this implies that s is rational, which will finish the problem. We know there exists rational numbers b and c such that $\lfloor a_n \rfloor = \lfloor bn + c \rfloor$. Let q be a positive integer such that qb is an integer, which exists since b is rational. Then we have that $\lfloor b(n+q) + c \rfloor = \lfloor bn + c \rfloor + bq$. Therefore,

$$\lfloor a_{n+q+1} \rfloor - \lfloor a_{n+q} \rfloor = \lfloor a_{n+1} \rfloor - \lfloor a_n \rfloor$$

for all n, and hence by (1) we get $d_{n+q} = d_n$ for all n. This means the sequence $d_1, d_2, d_3, \ldots$ is periodic, which means that s is rational. This completes the proof.

5/3/24. Let P and Q be two polynomials with real coefficients such that P has degree greater than 1 and

$$P(Q(x)) = P(P(x)) + P(x).$$

Show that $P(-x) = P(x) + x$.

Solution

Let n, m be the respective degrees of P, Q, and note that $n > 1$. From the given equation we have that the degree of $P(Q(x))$ is mn and the degree of $P(P(x)) + P(x)$ is n^2. Therefore $m = n$. Let

$$P(x) = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_0, \qquad Q(x) = q_n x^n + q_{n-1}x^{n-1} + \cdots + q_0.$$

Here p_n, q_n are nonzero. Consider the coefficient of x^{n^2} for each side of the given equation. Since $n > 1$, the $P(x)$ term of the given equation will not affect this coefficient, so we have $p_n \cdot p_n^n = p_n \cdot q_n^n$. Since these are real numbers, $p_n = \pm q_n$. We break into two cases:

Case 1: $p_n = q_n$. We show by strong induction on k that $p_{n-k} = q_{n-k}$ for $0 \le k \le n-1$. The base case $k = 0$ was assumed for this case. Given it is true for all values less than a certain k, consider the coefficient of x^{n^2-k} in the given equation. Note that since $k \le n-1$ and $n > 1$, we have $n^2 - k > n$, so the $P(x)$ term will not affect this coefficient. We compute that

$$\begin{aligned}P(Q(x)) &= p_n(Q(x))^n + p_{n-1}(Q(x))^{n-1} + \cdots + p_0,\\ &= p_n(q_n x^n + q_{n-1}x^{n-1} + \cdots + q_0)^n + p_{n-1}(q_n x^n + q_{n-1}x^{n-1} + \cdots + q_0)^{n-1} + \cdots + p_0.\end{aligned}$$

Because $n^2 - k > n(n-1)$, the only term that will contribute to the x^{n^2-k} coefficient is the term $p_n(Q(x))^n$. We can expand this using the multinomial theorem, obtaining

$$p_n(Q(x))^n = \sum_{0 \le i_1, i_2, \ldots, i_n \le n} p_n q_{i_1} q_{i_2} \cdots q_{i_n} x^{i_1 + i_2 + \cdots + i_n}.$$

Let A_k be the coefficient of x^{n^2-k} in this expansion. Then we have

$$A_k = \sum_{i_1 + i_2 + \cdots + i_n = n^2 - k} p_n q_{i_1} q_{i_2} \cdots q_{i_n}.$$

Notice that the only time any terms on the right side are not from the set $q_n, q_{n-1}, \ldots, q_{n-k+1}$ are when $n-1$ of the i_j are equal to n and the last one is equal to $n-k$. In all other cases, all terms in the sum of the i_j must be at least $n-k+1$.

Let B_k be the x^{n^2-k} coefficient of $P(P(x))$. By applying the same method of reasoning to $P(P(x))$ that we used for $P(Q(x))$, we find that

$$B_k = \sum_{i_1 + i_2 + \cdots + i_n = n^2 - k} p_n p_{i_1} p_{i_2} \cdots p_{i_n}.$$

But $A_k = B_k$. Furthermore, since $p_{n-j} = q_{n-j}$ for all $0 \le j < k$, when we set A_k and B_k equal, all terms of the sum will cancel except those terms where $n-1$ of the i_j equal n and the remaining i_j is equal to $n-k$. Accounting for only these terms of the sum, we get

$$np_n q_n^{n-1} q_{n-k} = np_n^n p_{n-k},$$

Cancelling the $np_n q_n^{n-1}$ from both sides using $p_n = q_n$, we are left with $q_{n-k} = p_{n-k}$, completing the inductive step.

Since the only coefficient that is not necessarily equal in $P(x)$ and $Q(x)$ is the constant term, we have $Q(x) - P(x) = a$ for some real number a. We can verify $P(x) = Q(x)$ cannot satisfy the given equation for nonzero polynomials, so $a \neq 0$. The given equation becomes $P(P(x) + a) = P(x) + P(P(x))$. Then letting $y = P(x)$, we have $P(y + a) - P(y) = y$. Since P is nonconstant, y can take values in an infinite set, so since this is an equality of polynomials, $P(y+a) - P(y) = y$ identically. The left side is a finite difference of P, so it has degree $n-1$ exactly. Since the right side has degree 1, $n = 2$, and thus $P(x) = p_2x^2 + p_1x + p_0$. Taking the linear coefficient of both sides of $P(x + a) - P(x) = x$, we get $2p_2a = 1$. Setting the constant term of $P(x + a) - P(x)$ equal to 0 and using $2p_2a = 1$, we get that $p_1 = -\frac{1}{2}$. Notice that $P(-x) - P(x) = -2p_1x = x$, so the required identity is satisfied in this case.

Case 2: $p_n = -q_n$. Observe that $Q(x) - P(x)$ divides $(Q(x))^k - (P(x))^k$ for any k. Therefore by taking an appropriate linear combination of terms of the form $(Q(x))^k - (P(x))^k$, we find that $Q(x) - P(x)$ divides $P(Q(x)) - P(P(x))$. By substituting the given identity in, we get that $Q(x) - P(x)$ divides $P(x)$, so $Q(x) - P(x)$ also divides $Q(x)$ by the Euclidean Algorithm. However, notice that $Q(x) - P(x)$ has degree n since the leading coefficient of the two polynomials is different. This means the same degree n polynomial divides $P(x)$ and $Q(x)$, both also degree n polynomials. As a result, $P(x)$ and $Q(x)$ are scalar multiples of each other. Since $p_n = -q_n$, we conclude that $P(x) = -Q(x)$.

Plugging this into the equation in the original problem gets $P(-P(x)) = P(P(x)) + P(x)$. Let $y = P(x)$ to obtain that $P(-y) = P(y) + y$. This identity is true for an infinite number of y since $P(x)$ takes values on an infinite set, so therefore it is true for all y, finishing this case.

Remark: Both of these cases admit solutions for P and Q. In case 1, we can have $P(x) = x^2 - \frac{x}{2}$ and $Q(x) = x^2 - \frac{x}{2} + \frac{1}{2}$. In case 2, we can have $P(x) = x^4 - \frac{x}{2}$ and $Q(x) = -x^4 + \frac{x}{2}$. Both of these pairs will satisfy $P(Q(x)) = P(P(x)) + P(x)$.