USA Mathematical Talent Search

Round 2 Solutions

Year 24 — Academic Year 2012–2013

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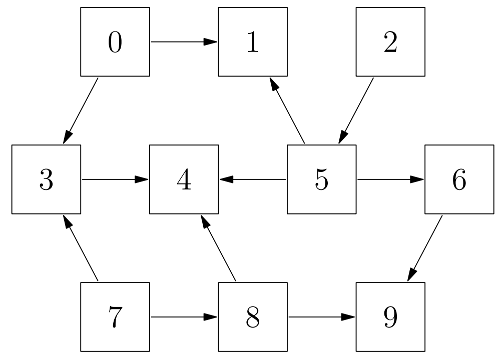
1/2/24. Fill in each of the ten boxes with a number so that the following conditions are satisfied.



1. Every number has three distinct digits that sum to 15. 0 may not be a leading digit. One digit of each number has been given to you.

2. No two numbers in any pair of boxes have the same unordered set of three digits. For example, it is not allowed for two different boxes to have the numbers 456 and 645.

3. Two boxes joined by an arrow must have two numbers which share an equal hundreds digit, tens digit, or ones digit. Also, the smaller number must point to the larger.

You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

To begin, notice that there are exactly 10 sets of three distinct unordered digits that sum to 15:

$$\{0,6,9\} \quad \{0,7,8\} \quad \{1,5,9\} \quad \{1,6,8\} \quad \{2,4,9\}$$
$$\{2,5,8\} \quad \{2,6,7\} \quad \{3,4,8\} \quad \{3,5,7\} \quad \{4,5,6\}$$

Since we have ten boxes to fill and each box must use a different set of digits, each of the digit sets above must be used for exactly one box.

Call the box with the digit d given the *d-box*. We know the 0-box either contains the digits $\{0,6,9\}$ or $\{0,7,8\}$. However, the 0-box has an arrow to the 3-box, so one of the digits in the 0-box is also contained in a number using the digit 3. None of 0, 6, and 9 can be contained as a digit in a box using the number 3. Therefore, the 0-box cannot use $\{0,6,9\}$ as its digit set, and must use $\{0,7,8\}$ instead.

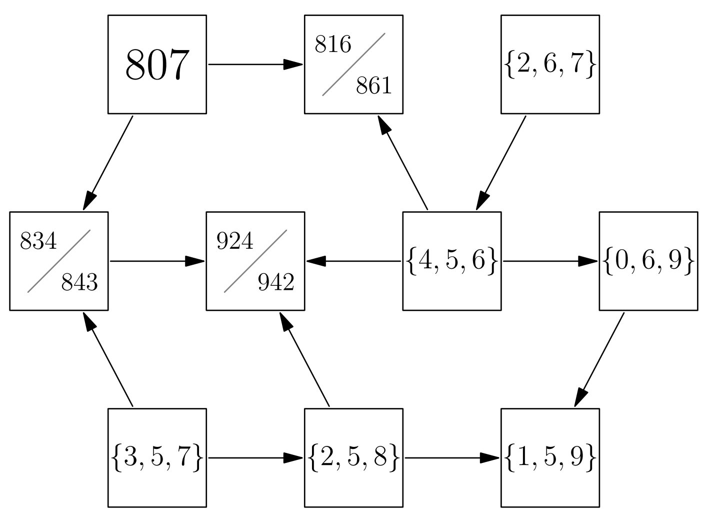
Now consider the digit shared between the 0-box and 1-box. The only possible sets for the 1-box are $\{1,5,9\}$ and $\{1,6,8\}$. Only $\{1,6,8\}$ shares a digit with the 0-box's set $\{0,7,8\}$, so the 1-box's digit set is $\{1,6,8\}$. Additionally, we also know that the 1-box has the greater number. Since the 0-box's number can't begin with 0, it is at least 700. The only digit the 1-box has that is at least 7 is 8, so 8 must be the hundreds digit. As 8 is the shared digit between the 0-box and 1-box, the 0-box also has 8 as the hundreds digit. So the 0-box is either 807 or 870. The latter would be greater than anything the 1-box can be, so the 0-box is 807.

We also have that the 3-box must have a number greater than 807. It can't have the digit 9, so its hundreds digit must be 8, and that is also the shared digit with the 0-box. The 3-box is either 834 or 843.

We now find the digit sets of the remaining boxes:

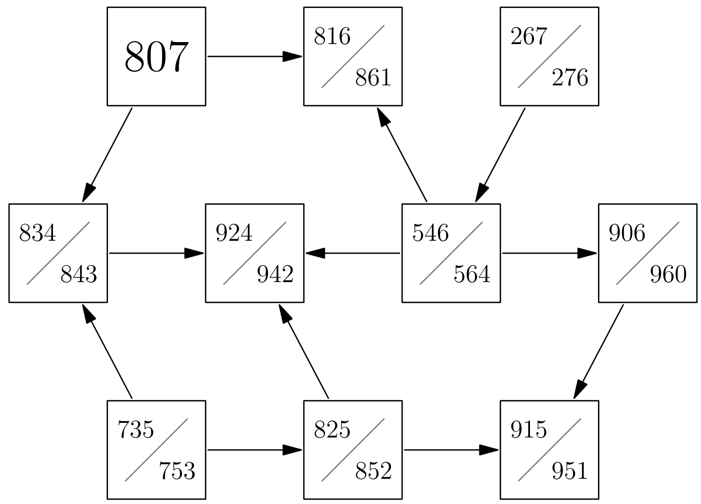
- Consider the 8-box. The only remaining digit set containing 8 is $\{2, 5, 8\}$, so this is the digit set of the 8-box.
- Consider the 7-box, which shares a digit with the sets $\{3, 4, 8\}$ and $\{2, 5, 8\}$. The only remaining set that contains 7 and one digit from each of these sets is $\{3, 5, 7\}$. Therefore box 7 has $\{3, 5, 7\}$ as its set.
- Consider the 4-box. It is larger than the 3-box's number that is at least 800. However all of the digit sets with 8 have been used up, so the 4-box must have 9 as its hundreds digit and thus the set $\{2, 4, 9\}$.
- Consider the 9-box. The only unused digit sets with 9 are $\{1, 5, 9\}$ and $\{0, 6, 9\}$. Only $\{1, 5, 9\}$ shares a digit with the 8-box's set $\{2, 5, 8\}$, so the 9-box's set is $\{1, 5, 9\}$.
- Consider the 5-box. The only unused digit set with 5 is $\{4, 5, 6\}$, so this the 5-box's set.
- Only the 2-box and 6-box remain, containing digit sets $\{2, 6, 7\}$ and $\{0, 6, 9\}$ between them. As only one has the digit 2, the 2-box must have $\{2, 6, 7\}$ and then the 6-box has $\{0, 6, 9\}$.

At this point we have determined the information shown in the image below.



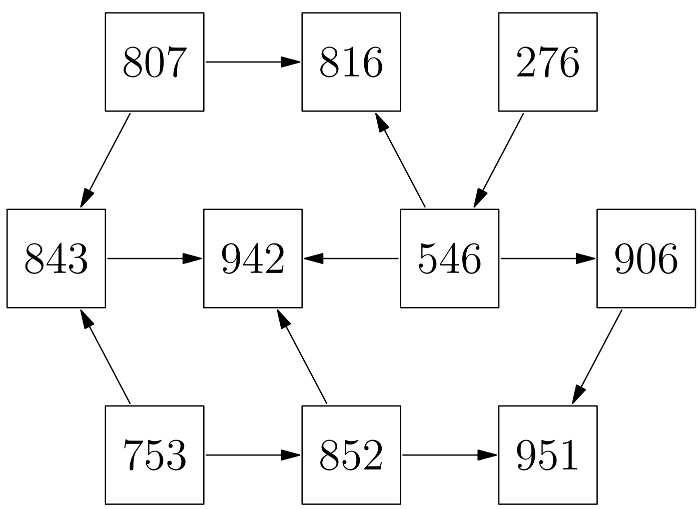
Next, we will determine the hundreds digit of each number. We already know the 0-box, 1-box, and 3-box all have hundreds digit 8, and the 4-box has hundreds digit 9. Consider the 5-box. It shares the digits 4 and 6 with the 4-box and 1-box respectively, neither of which are a hundreds digit. Therefore 5 is the hundreds digit. The 2-box, being smaller but having only one digit less than 5, must have 2 as its hundreds digit. In the 6-box, 0 is not allowed to be a hundreds digit and the 6 is shared with the 5-box, leaving 9 as the hundreds digit there. As it shares the 9 with the 9-box, the 9-box also has 9 as the hundreds digit. Finally, the 8-box shares the digits 2 and 5 with the 4-box and 9-box respectively, neither of which are hundreds digits, so the 8-box has 8 as its hundreds digit. The same logic applied to the 7-box shows it has 7 as the hundreds digit.

There are now only two choices left for each undetermined box. The image below summarizes these possibilities.



We are given that the 6-box, which is 906 or 960, is less than the 9-box, which is 915 or 951. Thus the 6-box is 906. From this the 5-box, 1-box, and 2-box, which all share the digit 6, must have 6 as their units digit and be 546, 816, and 276 respectively. Then the 4-box and 3-box, which both share the digit 4 with the 5-box, must have 4 as their tens digit and be 942 and 843 respectively. The 7-box is 753 since it shares the 3 with the 3-box. Finally, the 8-box and 9-box both share the digit 5 with the 7-box, so they are 852 and 951 respectively.

This gives the unique solution shown below. It can be verified that all of the constraints in the problem are satisfied by this configuration.



2/2/24. Find all triples (a, b, c) of positive integers with $a \le b \le c$ such that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) = 3.$$

Solution

Notice that since $a \le b \le c$,

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \le \left(1+\frac{1}{a}\right)^3,$$

so

$$\left(1+\frac{1}{a}\right)^3 \ge 3.$$

As a function in a, $\left(1+\frac{1}{a}\right)^3$ is decreasing. Also, $(1+\frac{1}{3})^3 = \frac{64}{27} < 3$, so $a \le 2$.

Case 1. $a = 1$. In this case, the given equation becomes

$$2\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) = 3.$$

This equation simplifies to $bc - 2b - 2c = 2$. By adding 4 to both sides and factoring, this equation becomes

$$(b-2)(c-2) = 6.$$

Since $1 \le b \le c$, we get $b - 2 = 1$ and $c - 2 = 6$, or $b - 2 = 2$ and $c - 2 = 3$. This leads to the solutions $(a, b, c) = (1, 3, 8)$ or $(1, 4, 5)$.

Case 2. $a = 2$. In this case, the given equation becomes

$$\frac{3}{2}\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) = 3.$$

This equation simplifies to $bc - b - c = 1$. By adding 1 to both sides and factoring, this equation becomes

$$(b-1)(c-1) = 2.$$

Since $2 \le b \le c$, we get $b-1 = 1$ and $c-1 = 2$. This leads to the solution $(a, b, c) = (2, 2, 3)$.

Hence, the three solutions are $(a, b, c) = (1, 3, 8), (1, 4, 5)$, or $(2, 2, 3)$.

3/2/24. Let $f(x) = x - \frac{1}{x}$, and define $f^1(x) = f(x)$, $f^n(x) = f(f^{n-1}(x))$ for $n \geq 2$. For each n, there is a minimal degree d_n such that there exist polynomials p and q with $f^n(x) = \frac{p(x)}{q(x)}$ and the degree of q is equal to d_n. Find d_n.

Solution

Let $f^n(x) = \frac{p_n(x)}{q_n(x)}$, where p_n and q_n do not share a common factor. We claim that for all $n \geq 1$,

$$(p_{n+1}(x), q_{n+1}(x)) = (p_n^2(x) - q_n^2(x), p_n(x)q_n(x)). \tag{1}$$

First, notice that

$$\frac{p_{n+1}(x)}{q_{n+1}(x)} = f^{n+1}(x) = f^n(f(x)) = f^n\left(x - \frac{1}{x}\right) = \frac{p_n(x)}{q_n(x)} - \frac{q_n(x)}{p_n(x)} = \frac{p_n^2(x) - q_n^2(x)}{p_n(x)q_n(x)}.$$

It remains to check that $p_n^2(x) - q_n^2(x)$ and $p_n(x)q_n(x)$ do not share a common factor. If they did, then $p_n^2(x) - q_n^2(x)$ must share a common factor $d(x)$ with either of $p_n(x)$ or $q_n(x)$. However, if $d(x)$ divides both $p_n^2(x) - q_n^2(x)$ and $p_n(x)$, this would imply that $d(x)$ also divides $-q_n^2(x)$. This implies the existence of a common factor dividing both $p_n(x)$ and $q_n(x)$, contradicting our definition that $p_n(x)$ and $q_n(x)$ have no common factor.

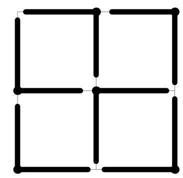
We can obtain a similar contradiction in the case that $d(x)$ divided both $p_n^2(x) - q_n^2(x)$ and $q_n(x)$. We conclude that $p_n^2(x) - q_n^2(x)$ and $p_n(x)q_n(x)$ do not have any common factors. This completes the proof of equation (1).

Next, we prove by induction on n that p_n has degree 2^n and q_n has degree $2^n - 1$ for all $n \geq 1$. For the base case $n = 1$, notice that $f^1(x) = \frac{x^2-1}{x}$, so $p_1(x) = x^2 - 1$ and $q_1(x) = x$. We see that p_n has degree $2^1 = 1$ and q_n has degree $2^1 - 1 = 1$, so the base case has been verified.

Now assume that for some particular value of n, p_n has degree 2^n and q_n has degree $2^n - 1$. By (1), we have $p_{n+1}(x) = p_n^2(x) - q_n^2(x)$. The first term has degree $2 \cdot 2^n = 2^{n+1}$ and the second term is of smaller degree, so the degree of the difference must be 2^{n+1}. Additionally, (1) gives that $q_{n+1} = p_n q_n$, so by the inductive hypothesis it has degree $2^n + (2^n - 1) = 2^{n+1} - 1$. This completes the induction.

Since d_n is equal to the degree of q_n, the answer to the original problem is $2^n - 1$.

4/2/24. Let n be a positive integer. Consider an $n \times n$ grid of unit squares. How many ways are there to partition the horizontal and vertical unit segments of the grid into $n(n+1)$ pairs so that the following properties are satisfied?



(i) Each pair consists of a horizontal segment and a vertical segment that share a common endpoint, and no segment is in more than one pair.

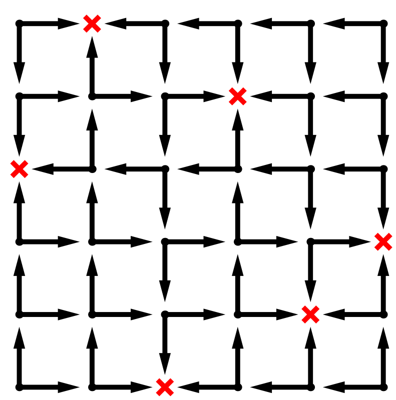
(ii) No two pairs of the partition contain four segments that all share the same endpoint.

(Pictured above is an example of a valid partition for $n = 2$.)

Solution

The grid contains $(n+1)^2$ vertices which are intersections of gridlines. Call such a vertex a *joint* if one of the pairs in the partition has this vertex as a common endpoint of its two segments. By condition (ii) no two pairs of the partition have the same joint. There are $n(n+1)$ pairs and $(n+1)^2$ vertices, so $n+1$ of the vertices are not joints. We will refer to these $n+1$ vertices as *isolated* vertices.

Suppose there is a row or column of the grid in which all $n+1$ vertices are joints. If it is a row, then each of the joints must use a horizontal segment of that row, but there are $n+1$ joints and only n segments, a contradiction. We get a similar contradiction for a column since there will be $n+1$ joints but only n vertical segments. Therefore each row and column must have at least one isolated vertex. Since there are $n+1$ rows and columns of vertices and $n+1$ total isolated vertices, each row and column has exactly one isolated vertex.



Call a set S of $n+1$ vertices in an $n \times n$ grid *allowable* if each row and column of vertices contains exactly one element of S. We claim that for every allowable set S, there exists a unique valid partition that has the vertices of S as its isolated vertices. Take an arbitrary endpoint P that is a joint. Its row and column both contain exactly one unused endpoint. Have P's horizontal segment "point" toward the unused endpoint of the row, meaning that if the unused endpoint is to the left of P, then P's horizontal segment should be to the left of P also. Similarly have P's vertical segment point toward the unused endpoint of the column. The image to the right shows a valid partition for $n = 5$ and has arrows drawn on the segments to clarify where they are pointing.

This construction will not have any segment of the grid be used by two endpoints. If a segment were reused, then it pointed in one direction for one of its endpoints and the other direction for the other endpoint, implying that there is an unused endpoint in both directions of the segment. This contradicts our earlier claim that no row or column contains two unused endpoints. Additionally, every segment of the grid will be used by this partition. This is because there are $2n(n+1)$ total horizontal and vertical segments of the grid, and we assigned 2 of them to each of $n(n+1)$ endpoints without any collisions.

Additionally, this construction is unique. A different construction would have required a joint P to have a segment which points away from the unused endpoint of the row or column. Call such a segment *bad*. Without loss of generality, suppose a joint P has a bad horizontal segment ℓ in a row. This bad segment points at another vertex of the row, which we will call Q. Q is a joint, so it is the shared endpoint of a horizontal segment. This segment must also point away from the isolated vertex of the row and thus be bad. We can continue this reasoning to find a sequence of bad segments until we eventually point at a vertex R on the edge of the grid. R is a joint, but there is no available horizontal segment that it is an endpoint of, a contradiction. Thus, all horizontal segments must point toward the isolated vertex of their row and all vertical segments must point toward the isolated vertex of their column, meaning the construction we proposed is the only one possible.

Since every allowable set S gives a unique valid partition with the vertices of S as isolated vertices, there is a 1-1 correspondence between allowable sets and valid partitions. It remains to count the number of allowable sets. Each row and column has exactly one vertex in S. There are $n+1$ choices for the vertex in the first row. In the second row, we can then include in S any of the n vertices that are not in the same column as the vertex chosen in the first row. Then the third row has $n-1$ choices for the vertex in S, since we cannot use the two vertices in the same column as those chosen from the previous two rows. This pattern continues, with $n+1-k$ ways to choose the vertex in the kth row. From this we obtain a total of $(n+1)\cdot n\cdot(n-1)\cdots 1=(n+1)!$ choices for an allowable set S. This is the number of valid partitions.

5/2/24. A unit square $ABCD$ is given in the plane, with O being the intersection of its diagonals. A ray ℓ is drawn from O. Let X be the unique point on ℓ such that $AX + CX = 2$, and let Y be the point on ℓ such that $BY + DY = 2$. Let Z be the midpoint of $\overline{XY}$, with $Z = X$ if X and Y coincide. Find, with proof, the minimum value of the length of OZ.

Solution

Place the points on the coordinate plane so that O is the origin, $A = \left(\frac{\sqrt{2}}{2}, 0\right)$, $B = \left(0, \frac{\sqrt{2}}{2}\right)$, $C = \left(\frac{-\sqrt{2}}{2}, 0\right)$, and $D = \left(0, -\frac{\sqrt{2}}{2}\right)$. Let θ be the real number in $[0, 2\pi)$ such that the ray ℓ makes an angle of θ with the x-axis, meaning that ℓ is the set of points $\{(r\cos\theta, r\sin\theta) \mid r \geq 0\}$.

The point X can be at any point on the plane whose sum of distances from the points A and C is 2. The set of such points is an ellipse with foci A and C. Let $\mathcal{E}_X$ be this ellipse. The center of the ellipse is O, since this is the midpoint of its foci A and C. The major axis of $\mathcal{E}_X$ is along the x-axis since both of its foci are on the x-axis, and the minor axis is along the y-axis since it is perpendicular to the major axis. Notice that since $AB + CB = 1 + 1 = 2$ and $AD + CD = 1 + 1 = 2$, B and D both lie on $\mathcal{E}_X$. Since the length of BD is $\sqrt{2}$, the length of the minor axis of $\mathcal{E}_X$ is $\sqrt{2}$. Since the sum of the distances to the foci is 2, the length of the major axis is 2. From the center of the ellipse and the length of its major and minor axes, we get that the equation for $\mathcal{E}_X$ is

$$x^2 + 2y^2 = 1.$$

The coordinates of the point X must satisfy this equation. We can do similar reasoning for set of possible positions for the point Y, finding that it forms an ellipse with foci B and D and center O, with the same lengths for the major and minor axes as $\mathcal{E}_Y$. Then the equation for $\mathcal{E}_Y$, which the coordinates for the point Y must satisfy, is

$$2x^2 + y^2 = 1.$$

Let r_X and r_Y be the positive real numbers such that $X = (r_X\cos\theta, r_X\sin\theta)$ and $Y = (r_Y\cos\theta, r_Y\sin\theta)$. Using our equations for $\mathcal{E}_X$ and $\mathcal{E}_Y$, we have

$$r_X^2(\cos^2\theta + 2\sin^2\theta) = 1,$$

$$r_Y^2(2\cos^2\theta + \sin^2\theta) = 1.$$

Simplifying, we get that $r_X = \frac{1}{\sqrt{1+\sin^2\theta}}$ and $r_Y = \frac{1}{\sqrt{1+\cos^2\theta}}$.

We are looking for the minimum length of OZ, which is equal to $\frac{r_X + r_Y}{2}$. Therefore, we need to minimize $r_X + r_Y$. Note that

$$\begin{aligned}
(r_X + r_Y)^2 &= \frac{1}{1+\sin^2\theta} + \frac{1}{1+\cos^2\theta} + \frac{2}{\sqrt{(1+\sin^2\theta)(1+\cos^2\theta)}} \\
&= \frac{3}{(1+\sin^2\theta)(1+\cos^2\theta)} + \frac{2}{\sqrt{(1+\sin^2\theta)(1+\cos^2\theta)}} \\
&= \frac{3}{2+\sin^2\theta\cos^2\theta} + \frac{2}{\sqrt{2+\sin^2\theta\cos^2\theta}} \\
&= \frac{3}{2+\frac{\sin^2(2\theta)}{4}} + \frac{2}{\sqrt{2+\frac{\sin^2(2\theta)}{4}}} \\
&= \frac{12}{8+\sin^2(2\theta)} + \frac{4}{\sqrt{8+\sin^2(2\theta)}}.
\end{aligned}$$

Thus it suffices to maximize $\sin^2(2\theta)$. Since $\sin^2(2\theta) \le 1$, we set $\sin^2(2\theta) = 1$. This occurs whenever θ makes an angle of 45° with one of the axes, meaning ℓ is parallel to a side of the square. For these values of θ, $(r_X + r_Y)^2 = \frac{8}{3}$, so $OZ = \frac{r_X + r_Y}{2} = \sqrt{\frac{2}{3}}$, and $\sqrt{\frac{2}{3}}$ is our answer.