USA Mathematical Talent Search
Round 1 Solutions
Year 24 — Academic Year 2012–2013
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1/1/24. Several children were playing in the ugly tree when suddenly they all fell.

- Roger hit branches A, B, and C in that order on the way down.
- Sue hit branches D, E, and F in that order on the way down.
- Gillian hit branches G, A, and C in that order on the way down.
- Marcellus hit branches B, D, and H in that order on the way down.
- Juan-Phillipe hit branches I, C, and E in that order on the way down.

Poor Mikey hit every branch A through I on the way down. Given only this information, in how many different orders could he have hit these 9 branches on the way down?

Let "$X < Y$" mean that Mikey must hit branch X before branch Y. Then the given data is:

$$\begin{aligned} A < B < C \qquad &(1),\\ D < E < F \qquad &(2),\\ G < A < C \qquad &(3),\\ B < D < H \qquad &(4),\\ I < C < E \qquad &(5). \end{aligned}$$

Combining (1), (3) and the second half of (5) gives

$$G < A < B < C < E \qquad (6),$$

and combining (2) and the first half of (4) gives

$$B < D < E < F \qquad (7).$$

Combining (6) and (7), we find that C and D must both be between B and E, but can appear in either order. So we must have *either* (8) or (9) below:

$$\begin{aligned} G < A < B < C < D < E < F \qquad &(8),\\ G < A < B < D < C < E < F \qquad &(9). \end{aligned}$$

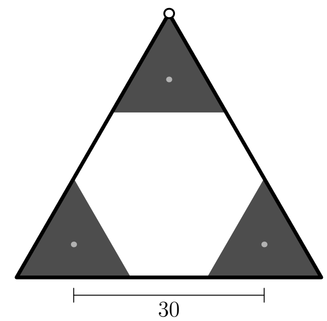
We only have to insert H and I into these chains. H must go after D, and I must go before C. We now break into cases based on whether (8) or (9) is true.

In case (8): We have 3 choices for where to insert H and 4 choices for where to insert I. These choices are independent, for a total of $3 \cdot 4 = 12$ possibilities.

In case (9): We have 4 choices for where to insert H and 5 choices for where to insert I. These choices are independent, but if we place both H and I between D and C, they can go in either order. Thus we have a total of $4 \cdot 5 + 1 = 21$ possibilities.

Thus, there are a total of $12 + 21 = \boxed{33}$ possibilities.

2/1/24. Three wooden equilateral triangles of side length 18 inches are placed on axles as shown in the diagram to the right. Each axle is 30 inches from the other two axles. A 144-inch leather band is wrapped around the wooden triangles, and a dot at the top corner is painted as shown. The three triangles are then rotated at the same speed and the band rotates without slipping or stretching. Compute the length of the path that the dot travels before it returns to its initial position at the top corner.



Let a third of a turn of the triangles be called a *thirn*, and assume the triangles are turning clockwise. Label the triangles A, B, C as in Figure 1. After each thirn, the dot's position will be exactly 18 inches farther along the perimeter of the band. Because the band is 144 inches long, it will take $144/18 = 8$ thirns for the dot to return to its original position. As this is an integer, the dot will return to its original position after only one complete rotation of the band.

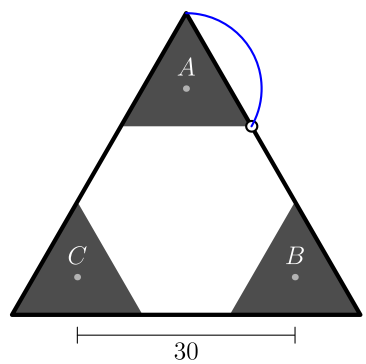


Figure 1

During the first thirn, the dot follows the path of the vertex of triangle A, tracing a third of the circumcircle of triangle A. The radius of this circle is the distance from center to the vertex of the triangular wheels, which is $18/\sqrt{3}$. Thus, the distance traveled by the point is $2\pi/3 \cdot 18/\sqrt{3} = 4\pi\sqrt{3}$ inches. (See Figure 1.)

During the second thirn, notice that the distance between the top corners of triangles A and B will remain 30 inches throughout the duration of the thirn. The 30 inch part of the band between these two corners will thus remain parallel to the sides of the original triangle, and each point along the band will be translated the same amount as the corners of the triangles A and B. Figure 2 shows this visually, with the paths of the triangle corners shown in red, a snapshot of the band's position drawn as a black segment between the red arcs, and the blue arc showing the path of the dot. Since the blue arc traces the same path shape as the corners of the triangles A and B, the dot moves the same distance as it did in the first thirn, which is $4\pi\sqrt{3}$ inches.

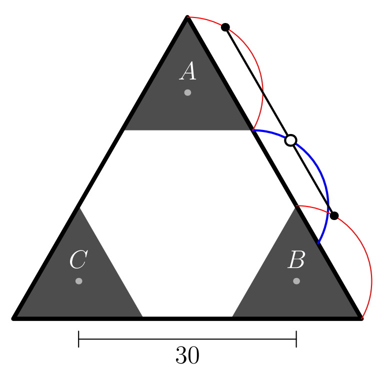


Figure 2

The dot ends the second thirn 6 inches away from the top corner of triangle B, which means it is also 6 inches from the center of triangle B. During the third thirn, the dot stays on triangle B, tracing a third of a circle with radius whose radius can be computed to be 6, traveling $2\pi/3 \cdot 6 = 4\pi$ inches. (See Figure 3.)

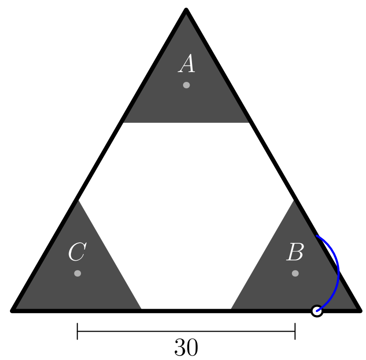


Figure 3

For reasoning identical to the second thirn, for the fourth thirn the dot also moves $4\pi\sqrt{3}$ inches, ending up in center of the bottom section of the band. This means that thirns 5 through 8 are a reflection of thirns 1 through 4, so the total distance traveled by the dot is $6 \cdot 4\pi\sqrt{3} + 2 \cdot 4\pi = \boxed{(24\sqrt{3}+8)\pi}$. Figure 4 shows the full path the dot travels.

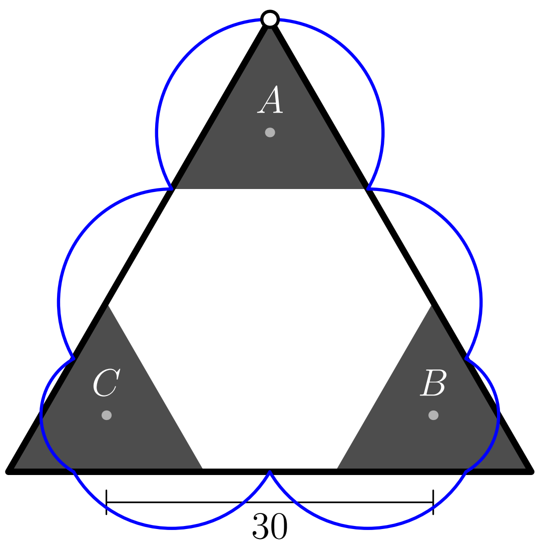


Figure 4

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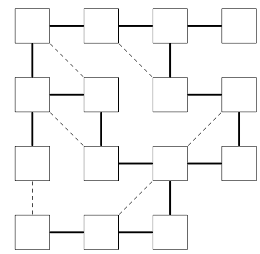
3/1/24. The **symmetric difference**, $\triangle$, of a pair of sets is the set of elements in exactly one set. For example,



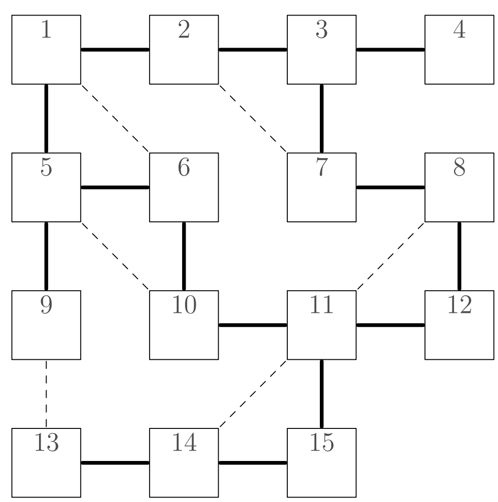
$$\{1, 2, 3\}\triangle\{2, 3, 4\} = \{1, 4\}.$$

There are fifteen nonempty subsets of $\{1, 2, 3, 4\}$. Assign each subset to exactly one of the squares in the grid to the right so that the following conditions are satisfied.

(i) If A and B are in squares connected by a solid line then $A\triangle B$ has exactly one element.

(ii) If A and B are in squares connected by a dashed line then the largest element of A is equal to the largest element of B.

You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)



The image to the left shows a numbering of the 15 boxes that we will refer to in this solution.

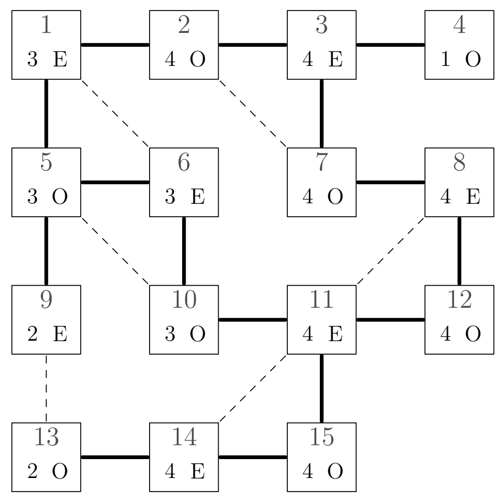
Before we begin, we note a certain logical configuration, which we will refer to as an *isosceles triangle*, in which we have three squares A, B, C where A, B are connected by a dashed line and C is connected by a solid line to each of A and B. An example of an isosceles triangle is $(A, B, C) = (8, 11, 12)$. In an isosceles triangle, not only do A and B have the same maximum element x, but C has that maximum element also. This is because if C had a different maximum y, then either $C\Delta A = C\Delta B = \{y\}$ (if $y > x$) or $C\Delta A = C\Delta B = \{x\}$ (if $y < x$). Either case implies $A = B$, contradicting the condition that all squares have a distinct subset.

By enumerating all of the 15 subsets, we can determine that there are:

- 8 subsets with largest element 4,
- 4 subsets with largest element 3,
- 2 subsets with largest element 2,
- 1 subset with largest element 1.

Note that squares 8, 11, and 14 must have the same largest element, and since they are part of isosceles triangles, 12 and 15 must have the same largest element as these three. So squares 8, 11, 12, 14, and 15 must all have the same largest element. There are not five subsets with largest element 1, 2, or 3, so largest element of squares 8, 11, 12, 14, and 15 must be 4.

Similarly, squares 1, 5, 6, and 10 all have the same largest element since $(1, 6, 5)$ and $(5, 10, 6)$ are isosceles triangles. If this largest element were 4, then nine subsets would have largest element 4, which is too many. On the other hand, there are not four subsets with largest element 1 or 2. So the largest element of squares 1, 5, 6, and 10 is 3. Squares 2, 3, and 7 also form an isosceles triangle and have the same largest element. We know where all sets with largest element 3 are, and this is too many subsets have to have largest element 1 or 2. So squares 2, 3, and 7 must have largest element 4. Out of the three remaining squares, 9 and 13 have a dashed line, so they have largest element 2, and 4 has largest element 1.



Finally, note that a solid line always joins a set with an odd number of elements to a set with an even number of elements, and that seven subsets have an even number of elements. Therefore, squares 1, 3, 6, 8, 9, 11, 14 have an even number of elements, and the other squares have an odd number of elements.

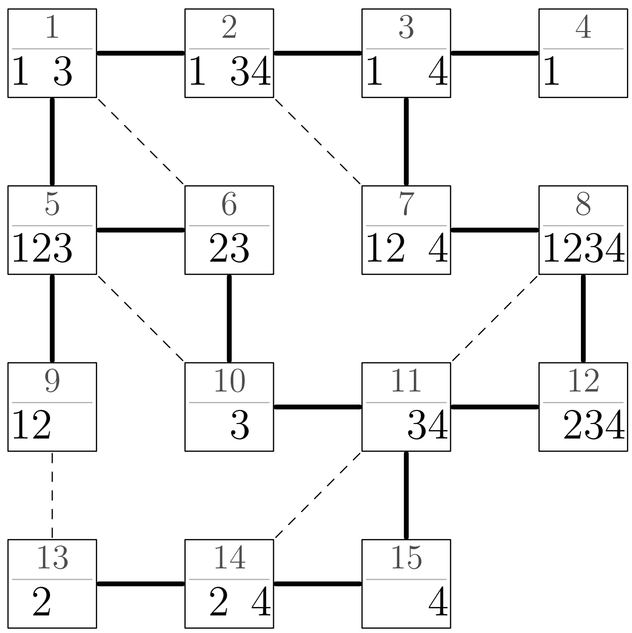
The diagram shown collects all the information we have so far. A number in a box is the largest element, an "E" means the box's subset has an even number of elements, and an "O" means it has an odd number of elements.

The information in the diagram will drive the rest of our progress. We immediately get square 4 is $\{1\}$. Since we know square 9 has an even number of elements, it has $\{1, 2\}$, while square 13 has $\{2\}$. These three squares are all connected to ones with a higher maximum element, so that maximum element must be the one added, and squares 3, 5, 14 have sets $\{1, 4\}$, $\{1, 2, 3\}$, and $\{2, 4\}$ respectively.

Square 10 is the only other square with maximum element 3 and an odd number of elements, so it has $\{3\}$. Then square 11, which has max element 4, is $\{3, 4\}$. The only square with maximum element 4 and an even number of elements we have yet to determine is square 8; it must have $\{1, 2, 3, 4\}$.

Now consider the possible locations of $\{4\}$. It can't touch $\{1, 2, 3, 4\}$ eliminating squares 7 and 12. If square 2 has $\{4\}$, then there is no way for square 1 to have maximum element 3. Therefore, square 15 has $\{4\}$. Likewise, $\{2, 3, 4\}$ must go in one of 2, 7, and 12 but can't touch $\{1, 4\}$, so it goes in square 12. This tells us that squares 2 and 7 must contain $\{1, 2, 4\}$ and $\{1, 3, 4\}$ in some order.

Since square 1 has largest element 3 and not 4, the edge from square 2 to square 1 must remove a 4, so square 2 has $\{1,3,4\}$ and square 1 has $\{1,3\}$. Then square 6, the last square with maximum element 3 has $\{2,3\}$. From our determining of square 2, we also now know that square 7 is $\{1,2,4\}$. This is the last box we needed to determine. The unique solution we have found is shown below.



4/1/24. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Let m be a positive integer, $m \geq 3$. For every integer i with $1 \leq i \leq m$, let

$$S_{m,i} = \left\{ \left\lfloor \frac{2^m - 1}{2^{i-1}} n - 2^{m-i} + 1 \right\rfloor : n = 1, 2, 3, \dots \right\}.$$

For example, for $m = 3$,

$$\begin{aligned} S_{3,1} &= \{\lfloor 7n - 3 \rfloor : n = 1, 2, 3, \dots\} \\ &= \{4, 11, 18, \dots\}, \\ S_{3,2} &= \left\{ \left\lfloor \frac{7}{2} n - 1 \right\rfloor : n = 1, 2, 3, \dots \right\} \\ &= \{2, 6, 9, \dots\}, \\ S_{3,3} &= \left\{ \left\lfloor \frac{7}{4} n \right\rfloor : n = 1, 2, 3, \dots \right\} \\ &= \{1, 3, 5, \dots\}. \end{aligned}$$

Prove that for all $m \geq 3$, each positive integer occurs in exactly one of the sets $S_{m,i}$.

Define

$$f(m, i, n) = \frac{2^m - 1}{2^{i-1}} n - 2^{m-i} + 1.$$

so that $S_{m,i} = \{\lfloor f(m, i, n) \rfloor : n = 1, 2, 3, \dots\}$. Let

$$T_{m,i} = \{\lfloor f(m, i, n) \rfloor : n = 1, 2, 3, \dots, 2^{i-1}\}.$$

We claim that $T_{m,i}$ is the set of numbers less than 2^m that end with exactly $m - i$ zeroes when written in binary. Rewrite $f(m, i, n)$ as

$$f(m, i, n) = \frac{2^m - 1}{2^{i-1}} n - 2^{m-i} + 1 = 2^{m-i}(2n - 1) + \frac{2^{i-1} - n}{2^{i-1}}.$$

On the right side the first term is an integer, and since $1 \leq n \leq 2^{i-1}$ the second term is at least 0 and less than 1. Therefore the floor of the expression is equal to the first term, meaning that $\lfloor f(m, i, n) \rfloor = 2^{m-i}(2n - 1)$.

A number ends in exactly $m - i$ zeroes if and only if it is an odd multiple of 2^{m-i}, and we see above that $\lfloor f(m, i, n) \rfloor$ ranges over the first 2^{i-1} odd multiples of 2^{m-i} as n ranges from 1 to 2^{i-1}. The 2^{i-1}th odd multiple is $2^m - 2^{m-i}$, the last one less than 2^m. This proves our claim that $T_{m,i}$ is the set of numbers less than 2^m which end with exactly $m - i$ 0s when written in binary.

Now notice that

$$\begin{aligned}f(m,i,n+2^{i-1}) &= \frac{2^m-1}{2^{i-1}}(n+2^{i-1}-2^{m-i}+1\\ &= \frac{2^m-1}{2^{i-1}}n-2^{m-i}+2^m\\ &= f(m,i,n)+(2^m-1).\end{aligned}$$

Therefore,

$$S_{m,i} = \{t+k(2^m-1) : t \in T_{m,i}, k=1,2,3,\ldots\}.$$

This means a number is in $S_{m,i}$ if and only if its remainder when divided by 2^m-1 is in $T_{m,i}$ (where we treat a remainder of 0 as a remainder of 2^m-1). Since each number from 1 to 2^m-1 occurs in exactly one of the $T_{m,i}$, this means every positive integer occurs in exactly one of the $S_{m,i}$, as desired.

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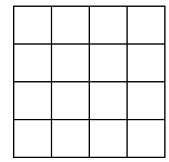
5/1/24. An ordered quadruple (y_1, y_2, y_3, y_4) is **quadratic** if there exist real numbers a, b, and c such that



$$y_n = an^2 + bn + c$$

for $n = 1, 2, 3, 4$.

Prove that if 16 numbers are placed in a 4×4 grid such that all four rows are quadratic and the first three columns are also quadratic then the fourth column must also be quadratic.

(We say that a row is quadratic if its entries, in order, are quadratic. We say the same for a column.)

We start with an important lemma.

Lemma: (y_1, y_2, y_3, y_4) is quadratic if and only if $y_4 - 3y_3 + 3y_2 - y_1 = 0$.

Proof: Suppose (y_1, y_2, y_3, y_4) is quadratic via the quadratic polynomial

$$f(x) = ax^2 + bx + c$$

so that $y_n = f(n)$ for $n = 1, 2, 3, 4$. Note that for all n,

$$f(n+1) - f(n) = a(n+1)^2 + b(n+1) + c - (an^2 + bn + c) = 2an + a + b.$$

Let $g(n) = 2an + a + b$. Then

$$g(n+1) - g(n) = 2a(n+1) + a + b - (2an + a + b) = 2a.$$

Plugging in $n = 3$ gives

$$2a = g(3) - g(2) = (f(3) - f(2)) - (f(2) - f(1)) = y_3 - 2y_2 + y_1, \qquad (1)$$

and plugging in $n = 4$ gives

$$2a = g(4) - g(3) = (f(4) - f(3)) - (f(3) - f(2)) = y_4 - 2y_3 + y_2. \qquad (2)$$

Thus $y_3 - 2y_2 + y_1 = y_4 - 2y_3 + y_2$, hence $y_4 - 3y_3 + 3y_2 - y_1 = 0$.

Conversely, it is well-known given any triple (y_1, y_2, y_3) we can find a quadratic $f(x) = ax^2 + bx + c$ such that $f(n) = y_n$ for $n = 1, 2, 3$. Then equations (1) and (2) tell us that $f(4) - 3y_3 + 3y_2 - y_1 = 0$. So we must have $f(4) = y_4$, meaning (y_1, y_2, y_3, y_4) is quadratic. $\square$

Let $a_{i,j}$ be the number in the ith row and jth column for $1 \le i, j \le 4$. Define for each $i = 1, 2, 3, 4$:

$$r_i = a_{i,1} - 3a_{i,2} + 3a_{i,3} - a_{i,4}, \qquad c_i = a_{1,i} - 3a_{2,i} + 3a_{3,i} - a_{4,i}.$$

By the lemma, row i is quadratic if and only if $r_i = 0$, and column i is quadratic if and only if $c_i = 0$. We are given that $r_i = 0$ for all i and $c_i = 0$ for $i \in \{1, 2, 3\}$, and we wish to show $c_4 = 0$.

Define the two quantities

$$R = r_1 - 3r_2 + 3r_3 - r_4, \qquad C = c_1 - 3c_2 + 3c_3 - c_4.$$

Suppose we expand out each of these sums in terms of the $a_{i,j}$. We find that

$$\begin{aligned} R = {} & a_{1,1} - 3a_{1,2} + 3a_{1,3} - a_{1,4} \\ & - 3a_{2,1} + 9a_{2,2} - 9a_{2,3} + 3a_{2,4} \\ & + 3a_{3,1} - 9a_{3,2} + 9a_{3,3} - 3a_{3,4} \\ & - a_{1,1} + 3a_{1,2} - 3a_{1,3} + a_{1,4} = C \end{aligned}$$

(The coefficient in front of $a_{i,j}$ can be written explicitly as $\binom{3}{i-1}\binom{3}{j-1}(-1)^{i+j}$. Notice that this is symmetric in i and j; this is the reason behind R and C giving the same expression.)

Thus, we have learned that $R = C$. Since $r_1 = r_2 = r_3 = r_4 = 0$ and $c_1 = c_2 = c_3 = 0$, we have from the original definitions of R and C that $R = 0$ and $C = -c_4$. This gives us $c_4 = 0$, as desired.

Remark: The ideas in this solution can be used to solve a generalized version of this problem, where the grid is $n \times n$ and the values of n rows and $n - 1$ columns are the first n values of a polynomial of degree at most $n - 2$.