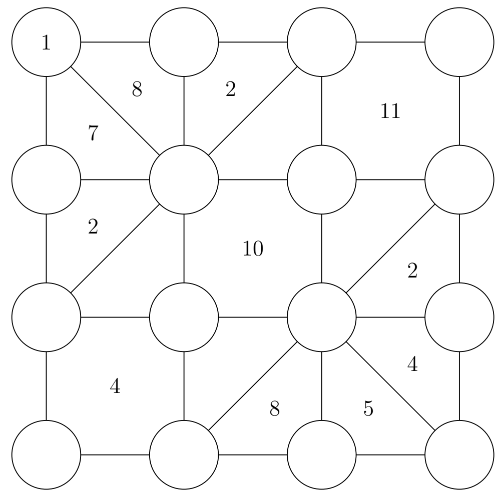
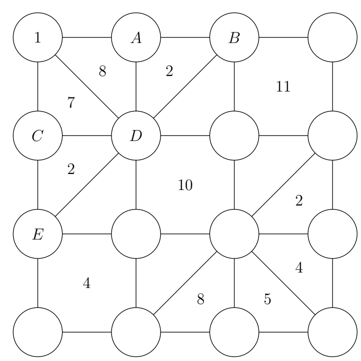
1/3/23. Fill in the circles to the right with the numbers 1 through 16 so that each number is used once (the number 1 has been filled in already). The number in any non-circular region is equal to the greatest difference between any two numbers in the circles on that region's vertices.

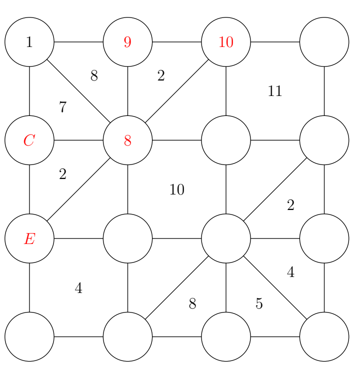
You do not need to prove that your configuration is the only one possible; you merely need to find a valid configuration. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)



First, we point out that if a triangular region contains the number 2, then the three vertices for the region must contain three consecutive numbers. In particular, if we look at the top-left region of the diagram, as in the figure to the right, then we see that both $\{A, B, D\}$ and $\{C, D, E\}$ must be sets of consecutive numbers, in some order. Since the greatest difference between two elements in the set $\{1, A, D\}$ is 8, and the smallest element is 1, A or D must be equal to 9.



Also, the greatest difference between two elements in the set $\{1, C, D\}$ is 7, so D cannot be 9, which means $A = 9$ and $D \leq 8$. Also, C or D must be equal to 8. If $C = 8$, then $D \leq 7$, but then $\{A, B, D\}$ cannot contain three consecutive numbers, so we must have $D = 8$. We know the set $\{C, D, E\} = \{C, 8, E\}$ must contain three consecutive numbers, and none of the elements are 9, so $\{C, E\} = \{6, 7\}$ in some order. The set $\{A, B, D\} = \{9, B, 8\}$ contains three consecutive numbers, and B cannot be 7, so $B = 10$.

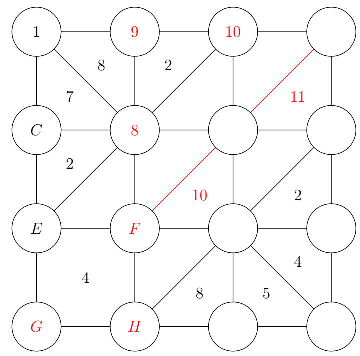


Now we have the figure to the left, where C and E are 6 and 7 in some order. We will call a number **small** if it is in the set $\{2, 3, 4, 5\}$, and we will call a number **large** if it is in the set

$$\{11, 12, 13, 14, 15, 16\}.$$

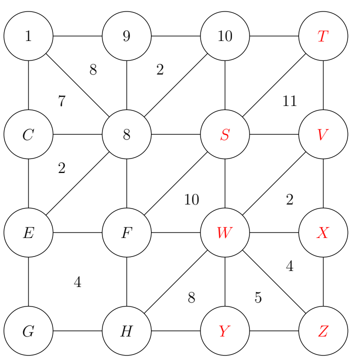
The unlabeled circles contain exactly those numbers that are either small or large. There are 4 small numbers and 6 large numbers.

Consider the numbers in the four vertices of the bottom-left square containing the number 4, including E. Since E is at most 7, and all these numbers are within 4 of each other, all three of the remaining numbers must be small. There is one more small number elsewhere in the grid. Now consider the numbers in the four vertices of the top-right square containing the number 11. Two of these numbers differ by 11, so they cannot be only 10 and large numbers. Hence, one of these numbers must be small. Also, the difference between two of these numbers is 11, but neither of these numbers can be the 10. Thus, we can replace the square containing the number 11 with a triangle containing the number 11, as shown below.



We can do the same for the center square containing the number 10. Among the four numbers attached to this square, two of them differ by 10. Neither of these numbers can be the 8, so we can replace the square containing the number 10 with a triangle containing the number 10, as shown to the right. We let F, G, and H be the three small numbers in the lower-left.

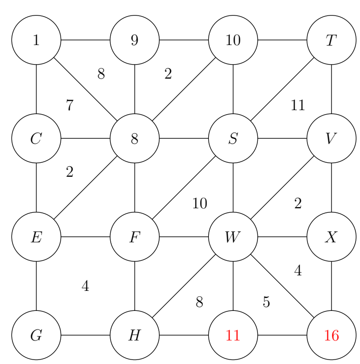
Next, consider the triangles labeled 2, 4, and 5 in the right side of the figure. Since the smallest difference between a small number and a large number is $11 - 5 = 6$, the five numbers attached to these triangles are either all large or all small. Since there are only four small numbers, all five of these numbers must be large. Let these five numbers be V, W, X, Y, and Z, as in the figure below. The final two vertices contain the one remaining small number and one remaining large number. We label these two vertices S and T.



We have the following:

$$\{C, E\} = \{6, 7\}$$
$$\{F, G, H\} \subset \{2, 3, 4, 5\}$$
$$\{V, W, X, Y, Z\} \subset \{11, 12, 13, 14, 15, 16\}$$

Either S is small and T is big, or T is small and S is big.

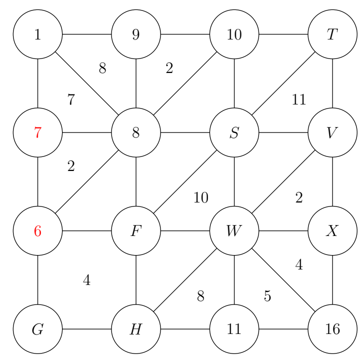


Consider the numbers W, Y, and Z. We know that the difference between two of these numbers is 5. But these numbers are all large, and the only way that the difference between two large numbers is 5 is if the numbers are 11 and 16. Suppose W or Y is equal to 16. We know that the greatest difference among the numbers H, W, and Y is 8, which means H must be equal to 8. However, we have already used 8, so neither W nor Y can be 16, which means $Z = 16$. Then W or Y is equal to 11. We know that the greatest difference among the numbers X, W, and Z (which is 16) is 4, so W cannot be 11, which means $Y = 11$.

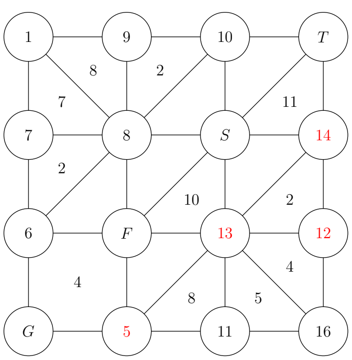
It also follows that W or X is equal to 12. We know that the greatest difference among the numbers V, W, and X is 2, so these three numbers are consecutive. Also, all of these numbers are large, and we have already used 11, so these numbers are 12, 13, and 14, in some order. Therefore, the remaining large number must be 15. We know that one of M and N is the remaining large number, and the other is the remaining small number. Furthermore, these numbers differ by 11, so the remaining small number is $15 - 11 = 4$.

Hence, the small numbers F, G, and H are 2, 3, and 5, in some order. We know that the greatest difference among the numbers E, F, G, and H is 4. But the largest number is E, and the smallest number is 2, so $E - 2 = 4$, which means $E = 6$. Then $C = 7$.

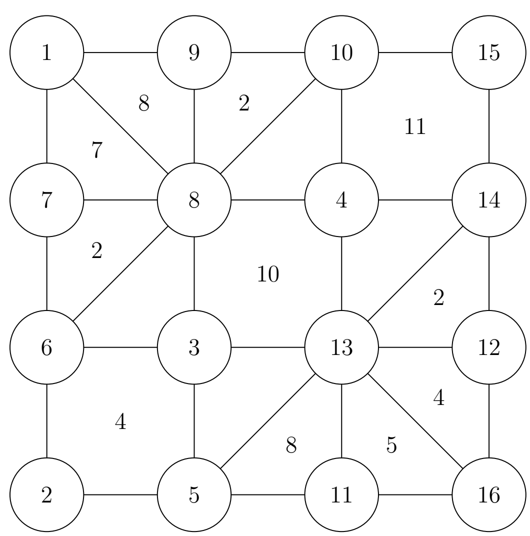
Now consider the numbers W, H, and $Y = 11$. We know that the greatest difference among these numbers is 8. Since W is either 12, 13, or 14, and H is 2, 3, or 5, the greatest difference among these numbers is $W - H$. Furthermore, the only values that satisfy $W - H = 8$ are $W = 13$ and $H = 5$.



Then X is 12 or 14. But the greatest difference among the numbers $W = 13$, X, and $Z = 16$ is 4, so $X = 12$, which makes $V = 14$.



Finally, F and G are 2 and 3 in some order, and M and N are 4 and 15 in some order. The greatest difference among F, N, and $W = 13$ is 10. By a quick check, the only values for which this occurs are $F = 3$ and $N = 4$. Then $G = 2$ and $M = 15$, which completes the grid.



2/3/23. Let x be a complex number such that $x^{2011} = 1$ and $x \neq 1$. Compute the sum

$$\frac{x^2}{x-1} + \frac{x^4}{x^2-1} + \frac{x^6}{x^3-1} + \cdots + \frac{x^{4020}}{x^{2010}-1}.$$

Let S denote the given sum, so

$$S = \frac{x^2}{x-1} + \frac{x^4}{x^2-1} + \cdots + \frac{x^{4020}}{x^{2010}-1} = \sum_{k=1}^{2010} \frac{x^{2k}}{x^k - 1}. \tag{1}$$

We can reverse the order of the terms, to get

$$S = \frac{x^{4020}}{x^{2010}-1} + \frac{x^{4018}}{x^{2009}-1} + \cdots + \frac{x^2}{x-1} = \sum_{k=1}^{2010} \frac{x^{4022-2k}}{x^{2011-k}-1}.$$

Since $x^{2011} = 1$,

$$\frac{x^{4022-2k}}{x^{2011-k}-1} = \frac{x^{-2k}}{x^{-k}-1} = \frac{1}{x^k - x^{2k}} = \frac{1}{x^k(1-x^k)},$$

so

$$S = \sum_{k=1}^{2010} \frac{1}{x^k(1-x^k)}. \tag{2}$$

Adding equations (1) and (2), we get

$$\begin{aligned}
2S &= \sum_{k=1}^{2010} \frac{x^{2k}}{x^k-1} + \sum_{k=1}^{2010} \frac{1}{x^k(1-x^k)} \\
&= \sum_{k=1}^{2010} \left[\frac{x^{2k}}{x^k-1} + \frac{1}{x^k(1-x^k)} \right] \\
&= \sum_{k=1}^{2010} \left[\frac{x^{3k}}{x^k(x^k-1)} - \frac{1}{x^k(x^k-1)} \right] \\
&= \sum_{k=1}^{2010} \frac{x^{3k}-1}{x^k(x^k-1)}.
\end{aligned}$$

We can factor $x^{3k}-1$ as $(x^k-1)(x^{2k}+x^k+1)$, so

$$\begin{aligned}
2S &= \sum_{k=1}^{2010} \frac{(x^k-1)(x^{2k}+x^k+1)}{x^k(x^k-1)} \\
&= \sum_{k=1}^{2010} \frac{x^{2k}+x^k+1}{x^k} \\
&= \sum_{k=1}^{2010} \left(x^k+1+\frac{1}{x^k}\right) \\
&= \left(x+1+\frac{1}{x}\right)+\left(x^2+1+\frac{1}{x^2}\right)+\cdots+\left(x^{2010}+1+\frac{1}{x^{2010}}\right) \\
&= (x+x^2+\cdots+x^{2010})+2010+\frac{1}{x}+\frac{1}{x^2}+\cdots+\frac{1}{x^{2010}}.
\end{aligned}$$

Since $x^{2011}=1$, we have that $x^{2011}-1=0$, which factors as

$$(x-1)(x^{2010}+x^{2009}+\cdots+x+1)=0.$$

We know that $x\neq 1$, so we can divide both sides by $x-1$, to get

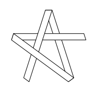
$$x^{2010}+x^{2009}+\cdots+x+1=0.$$

Then

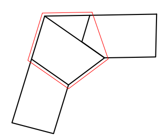
$$\begin{aligned}
2S &= (x+x^2+\cdots+x^{2010})+2010+\frac{1}{x}+\frac{1}{x^2}+\cdots+\frac{1}{x^{2010}} \\
&= (x+x^2+\cdots+x^{2010})+2010+\frac{x^{2010}+x^{2009}+\cdots+x}{x^{2011}} \\
&= (-1)+2010+\frac{-1}{1} \\
&= 2008,
\end{aligned}$$

so $S=1004$.

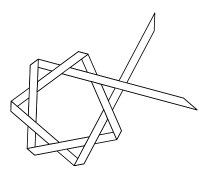
3/3/23. A long, 1-inch wide strip of cloth can be folded into the figure below.



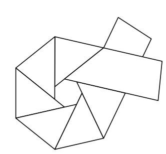
When the cloth is pulled tight and flattened, the result is a knot with two trailing strands. The knot has outer boundary equal to a regular pentagon as shown below.



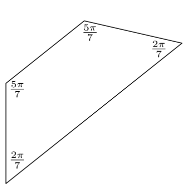
Instead, a long 1-inch wide strip of cloth is folded into the next figure, following the given turns and crossings.



When the cloth is pulled tight and flattened, the result is a knot with two trailing strands. The knot has outer boundary equal to a regular heptagon. The trailing strands of the heptagonal knot are both cut at the outer (heptagonal) boundary of the knot. Then the knot is untied. What is the area of one side of the resulting quadrilateral of cloth? (Your answer may contain trigonometric expressions.)

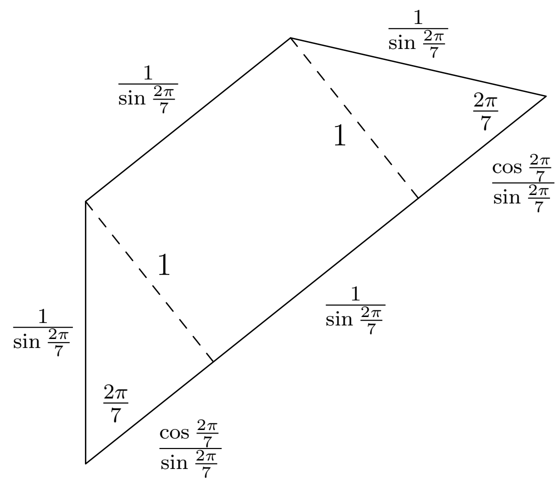


We are given that the knot, when pulled tight, forms a regular heptagon (with two trailing strands). This tells us that the angles at each crease will all be the interior angles of a heptagon, $\frac{5\pi}{7}$. The piece of cloth between two creases (when the strip is unfolded) is then forced to be an isosceles trapezoid with acute angles $\frac{2\pi}{7}$, and whose legs are equal to the short base. Furthermore, the height of each trapezoid is equal to the width of the cloth, namely 1 inch.



When the trailing strands are cut and the cloth is untied, the cloth becomes the union of the seven trapezoids. Our goal is now to compute the area of one of these trapezoids.

Each leg of the trapezoid is the hypotenuse of a right triangle with angle $\frac{2\pi}{7}$ and opposite side 1, so the length of each leg is $\frac{1}{\sin\frac{2\pi}{7}}$. This means that the short base has length $\frac{1}{\sin\frac{2\pi}{7}}$ as well.



Furthermore, the length of the long base is then

$$\frac{\cos\frac{2\pi}{7}}{\sin\frac{2\pi}{7}} + \frac{1}{\sin\frac{2\pi}{7}} + \frac{\cos\frac{2\pi}{7}}{\sin\frac{2\pi}{7}} = \frac{2\cos\frac{2\pi}{7}+1}{\sin\frac{2\pi}{7}}.$$

The height of the trapezoid is 1, so the area of the trapezoid is equal to the average of the lengths of the bases, which is

$$\frac{1}{2}\left(\frac{1}{\sin\frac{2\pi}{7}} + \frac{2\cos\frac{2\pi}{7}+1}{\sin\frac{2\pi}{7}}\right) = \frac{\cos\frac{2\pi}{7}+1}{\sin\frac{2\pi}{7}}.$$

Using the identity $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$, we can simplify this to

$$\frac{\cos\frac{2\pi}{7}+1}{\sin\frac{2\pi}{7}} = \cot\frac{\pi}{7}.$$

Finally, the entire strip consists of seven of these trapezoids, so the area of the strip, in square inches, is

$$7\cot\frac{\pi}{7}.$$

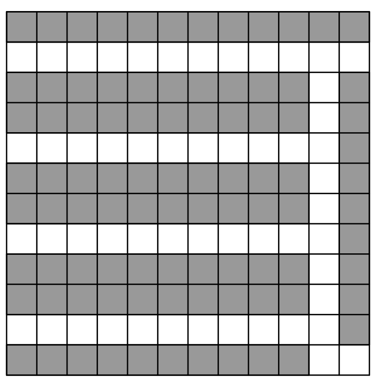
4/3/23. Renata the robot packs boxes in a warehouse. Each box is a cube of side length 1 foot. The warehouse floor is a square, 12 feet on each side, and is divided into a 12-by-12 grid of square tiles 1 foot on a side. Each tile can either support one box or be empty. The warehouse has exactly one door, which opens onto one of the corner tiles.

Renata fits on a tile and can roll between tiles that share a side. To access a box, Renata must be able to roll along a path of empty tiles starting at the door and ending at a tile sharing a side with that box.

(a) Show how Renata can pack 91 boxes into the warehouse and still be able to access any box.

(b) Show that Renata **cannot** pack 95 boxes into the warehouse and still be able to access any box.

(a) The following diagram shows that Renata can have access to 91 boxes.



(b) **Solution 1**: We say that a platform is *occupied* if it has a box on it, and that a platform is *reachable* if it is not occupied and Renata can roll to it from the door. Two platforms are *neighboring* if they share a side. We refer to platforms that have 4, 3, or 2 neighboring platforms as *interior*, *edge*, and *corner* platforms, respectively.

Suppose Renata has packed boxes in such a way that she can access any box. Let r_{I}, r_{E}, and r_{C} be respectively the number of interior, edge, and corner platforms that are reachable, and let $r = r_{\mathrm{I}} + r_{\mathrm{E}} + r_{\mathrm{C}}$ be the total number of reachable platforms. Let s be the number of unoccupied platforms that are unreachable. Thus, the number of boxes is $144 - r - s$.

Let N be the number of ordered pairs (x, y) of neighboring platforms such that y is reachable. We count N in two different ways. On one hand,

$$N = \sum_{\substack{\text{reachable} \\ \text{platforms } y}} (\#\text{ neighbors of } y),$$

which gives us $N = 4r_{\mathrm{I}} + 3r_{\mathrm{E}} + 2r_{\mathrm{C}}$.

On the other hand, we can write

$$N = \sum_{\substack{\text{all plat-} \\ \text{forms } x}} (\#\text{ reachable neighbors of } x),$$

then split the terms of this sum into two groups:

$$N = \sum_{\substack{\text{occupied} \\ \text{platforms } x}} (\#\text{ reachable neighbors of } x) + \sum_{\substack{\text{unoccupied} \\ \text{platforms } x}} (\#\text{ reachable neighbors of } x).$$

Each term in the first sum is at least 1, since every box must neighbor a reachable platform. Thus, the first sum is at least $144 - r - s$. To bound the second term, we note that if we think of the platforms as vertices of a graph with edges connecting neighboring platforms, then the reachable platforms form a connected subgraph. This subgraph has r vertices, so it has at least $r - 1$ edges, making the second sum greater than or equal to $2r - 2$ (since this sum counts pairs of neighboring reachable platforms **with order**).

Therefore, $N \geq (144 - r - s) + (2r - 2) \geq 142 + r - s$. Comparing this to our first way of counting N, we get

$$4r_{\mathrm{I}} + 3r_{\mathrm{E}} + 2r_{\mathrm{C}} \geq 142 + r - s.$$

Subtracting r from both sides gives

$$3r_{\mathrm{I}} + 2r_{\mathrm{E}} + r_{\mathrm{C}} \geq 142 - s,$$

and so

$$3r = 3r_{\mathrm{I}} + 3r_{\mathrm{E}} + 3r_{\mathrm{C}} \geq 142 - s + r_{\mathrm{E}} + 2r_{\mathrm{C}}.$$

In particular, we have

$$r + s = \frac{3r + 3s}{3} \geq \frac{142 + 2s + r_{\mathrm{E}} + 2r_{\mathrm{C}}}{3}.$$

Note that $r_{\mathrm{C}} \geq 1$, since the platform with the door is reachable, and $r_{\mathrm{E}} + 2s \geq 4$, since every corner must either have a reachable neighbor or be unoccupied and unreachable. Thus,

$$r + s \geq \frac{148}{3} > 49.$$

Since $r + s$ is the number of unoccupied platforms, we know there are at least 50 platforms without boxes, and thus no more than 94 with boxes.

Solution 2: Let $f(n)$ be the maximum number of boxes that Renata can reach, where n is the number of tiles that she can roll onto. Note that any configuration of the first 3 squares of the path allows Renata to reach 3 boxes, so $f(3) = 4$. Each additional square added to the path allows an increase of at most 2 to the number of reachable boxes: it removes the box on the path's added square, but that added square potentially reaches 3 new boxes. So, $f(n) \leq 2n - 2$ for all $n \geq 3$. Noting that there are 144 tiles total, we have

$$f(n) \leq \min\{144 - n, 2n - 2\} = \begin{cases} 2n - 2 & \text{if } 3 \leq n \leq 48, \\ 144 - n & \text{if } n \geq 49. \end{cases}$$

The only positive integer n for which $f(n) \geq 95$ is $n = 49$, so to reach 95 boxes, we must have a 49-tile path. By the initial construction, such a path could optimally reach $2(49) - 2 = 96$ boxes. If we can lower this bound for a 49-tile path by 2, we will have shown that no path of length 49 can reach 95 or more boxes. Therefore, there exists no path that reaches 95 or more boxes.

In order for Renata to reach 95 boxes on a 49-tile path, since $95 + 49 = 144$, Renata's path must contain at least one neighbor of each of the other three corner tiles (either to access a box on that tile, or to reach that tile in her path). However, whenever Renata adds one of these three tiles that neighbor a corner to her path, she only increases the number of reachable squares by at most 1 (the wall of the warehouse taking the place of the second new reachable tile). Therefore, Renata can reach at most $96 - 3 = 93$ tiles from a path of length 49.

An alternative argument goes as follows. Every time the path branches (that is, a new square is added to the path anywhere other than at the end of the path), only at most 1 additional box is reached since at least 1 of the three "new" boxes reached by the new square was already reachable by the path). Every time the path turns a corner, only at most 1 additional box is reached since at least 1 of the three "new" boxes reached by the new square was again already reachable by the path. Because the warehouse is a 12×12 grid, a 49-square path must turn or branch at least twice after the first three tiles. Such a path can reach at most $96 - 2 = 94$ boxes, so a path that reaches 95 boxes is not possible.

5/3/23. Let $k > 2$ be a positive integer. Elise and Xavier play a game that has four steps, in this order.

1. Elise picks 2 nonzero digits (1-9), called e and f.
2. Xavier then picks k nonzero digits (1-9), called $x_1, \ldots, x_k$.
3. Elise picks any positive integer d.
4. Xavier picks an integer $b > 10$.

Each player's choices are known to the other player when the choices are made.

The winner is determined as follows. Elise writes down the two-digit base b number ef_b. Next, Xavier writes the k-digit base b number that is constructed by concatenating his digits,

$$(x_1 \ldots x_k)_b.$$

They then compute the greatest common divisor (gcd) of these two numbers. If this gcd is greater than or equal to the integer d then Xavier wins. Otherwise Elise wins.

(As an example game for $k = 3$, Elise chooses the digits $(e, f) = (2, 4)$, Xavier chooses $(4, 4, 8)$, and then Elise picks $d = 100$. Xavier picks base $b = 25$. The base-25 numbers 24_{25} and 448_{25} are, respectively, equal to 54 and 2608. The greatest common divisor of these two is 2, which is much less than 100, so Elise wins handily.)

Find all k for which Xavier can force a win, no matter how Elise plays.

We claim that Xavier has a winning strategy if and only if k is even.

Let k be an even integer. After Elise chooses her digits e and f, Xavier can set his digits to be $x_1 = e$, $x_2 = f$, $x_3 = e$, $x_4 = f$, $\ldots$, $x_{k-1} = e$, and $x_k = f$. Then Elise's number ef_b divides Xavier's number $efef \ldots ef_b$, so their greatest common divisor is simply Elise's number $ef_b = be + f$.

Then no matter what positive integer d Elise chooses, Xavier can choose an integer b sufficiently large so that $be + f \geq d$, and Xavier wins.

Now, let k be an odd integer. Elise can start by choosing $e = 1$ and $f = 9$. Let Xavier choose his digits x_1, x_2, $\ldots$, x_k. Let $F(t) = t + 9$ and

$$G(t) = x_1 t^{k-1} + x_2 t^{k-2} + \cdots + x_{k-1} t + x_k.$$

Note that $F(b)$ and $G(b)$ are equal to Elise's number and Xavier's number in base b, respectively. We will show there exists a constant M such that $\gcd(F(b), G(b)) < M$ for all integers $b > 10$.

When the polynomial $F(t)$ is divided into the polynomial $G(t)$, we obtain a quotient $Q(t)$ and remainder $R(t)$, so

$$G(t) = F(t)Q(t) + R(t).$$

Both $G(t)$ and $F(t)$ have integer coefficients, and the leading coefficient of $F(t)$ is 1, so both $Q(t)$ and $R(t)$ have integer coefficients as well. Furthermore, $F(t)$ is a linear polynomial, so the remainder $R(t)$ is a constant, say c.

We will show that this constant c is positive. Since $F(t) = t + 9$, we can set $t = -9$ in the equation above, to get

$$c = G(-9) = x_1(-9)^{k-1} + x_2(-9)^{k-2} + \cdots + x_{k-2}(-9)^2 + x_{k-1}(-9) + x_k.$$

Since k is odd, and each x_i is a digit from 1 to 9, this expression is minimized when $x_1 = x_3 = \cdots = x_k = 1$ and $x_2 = x_4 = \cdots = x_{k-1} = 9$. Hence,

$$\begin{aligned} c &= G(-9) \\ &= x_1(-9)^{k-1} + x_2(-9)^{k-2} + \cdots + x_{k-2}(-9)^2 + x_{k-1}(-9) + x_k \\ &\geq (-9)^{k-1} + 9 \cdot (-9)^{k-2} + (-9)^{k-3} + 9 \cdot (-9)^{k-4} + \cdots + (-9)^2 + 9 \cdot (-9) + 1 \\ &= 1. \end{aligned}$$

Therefore, c is positive.

Now, setting $t = b$ in the equation above, we get

$$G(b) = F(b)Q(b) + c.$$

Since $\gcd(F(b), G(b))$ divides both $F(b)$ and $G(b)$, $\gcd(F(b), G(b))$ also divides $G(b) - F(b)Q(b) = c$. In particular, $\gcd(F(b), G(b))$ must be less than or equal to c, for any integer $b > 10$.

Elise can then choose any integer d greater than c (say $d = c + 1$). Then no matter what base b Xavier chooses, $\gcd(F(b), G(b))$ will be less than or equal to c, so it will be less than d. Thus, Elise has a winning strategy when k is odd.

Credits: Problem 1/3/23 and 5/3/23 proposed by Palmer Mebane.
All other problems and solutions by USAMTS staff.