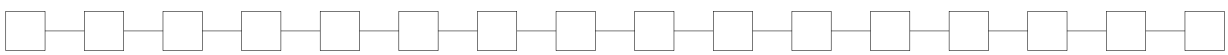
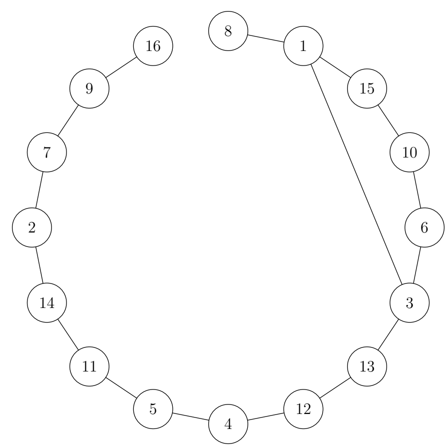
1/2/23.

Find all the ways of placing the integers 1, 2, 3, ..., 16 in the boxes below, such that each integer appears in exactly one box, and the sum of every pair of neighboring integers is a perfect square.



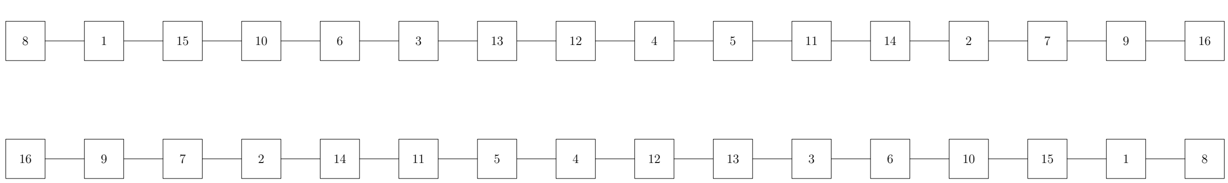
First, we construct a graph with 16 vertices, where each vertex corresponds to one of the integers from 1 through 16, and two vertices are joined by an edge if the sum of the integers corresponding to the vertices is a perfect square. This gives us the graph to the right.



A solution to this problem corresponds to a path in this graph that passes through every vertex exactly once (also known as a Hamiltonian path). The vertices corresponding to the integers 8 and 16 have only one edge, so the path must start at the vertex labelled 8 and end at the vertex labelled 16, or vice versa.

If the first number is 8, then the next number must be 1. We then have two choices for the number after 1, namely 3 and 15. If the number after 1 is 3, then the path cannot reach the vertex labelled 15 (because the last number must be 16), so the number after 1 must be 15, and then all numbers after that are uniquely determined. This gives us the first solution shown below.

If the first number is 16, then the next numbers must be 9, 7, and so on, until 3. We then have two choices for the number after 3, namely 1 and 6. If the number after 3 is 1, then the path cannot reach the vertex labelled 6 (because the last number must be 8), so the number after 3 must be 6, and then all numbers after that are uniquely determined. This gives us the second solution shown below.



2/2/23.

Four siblings are sitting down to eat some mashed potatoes for lunch: Ethan has 1 ounce of mashed potatoes, Macey has 2 ounces, Liana has 4 ounces, and Samuel has 8 ounces. This is not fair. A *blend* consists of choosing any two children at random, combining their plates of mashed potatoes, and then giving each of those two children half of the combination. After the children's father performs four blends consecutively, what is the probability that the four children will all have the same amount of mashed potatoes?

To see how all four children can end up with the same amount of mashed potatoes after four blends, we start with the final amounts and work backwards. The total amount of mashed potatoes at the start, in ounces, is $1 + 2 + 4 + 8 = 15$. This total never changes, so at the end, each child must have $\frac{15}{4}$ ounces of mashed potatoes.

We now consider the blends in reverse order. It is possible that the father kept blending after everyone reached equal amounts of $\frac{15}{4}$, but at some point, the father must have blended two different amounts to produce two equal amounts of $\frac{15}{4}$. Let these two different amounts be $\frac{15}{4}+x$ and $\frac{15}{4}-x$, where x is a positive real number. So at this point, the amounts among the children are $\frac{15}{4}$, $\frac{15}{4}$, $\frac{15}{4}+x$, and $\frac{15}{4}-x$.

We continue to consider the blends in reverse order. A blend always produces two equal amounts, and the only two equal amounts that we have are $\frac{15}{4}$ and $\frac{15}{4}$. Therefore, at some point, the father must have blended another two different amounts to produce two equal amounts of $\frac{15}{4}$. Let these two different amounts be $\frac{15}{4}+y$ and $\frac{15}{4}-y$, where y is a positive real number. At this point, the amounts among the children are $\frac{15}{4}+x$, $\frac{15}{4}-x$, $\frac{15}{4}+y$, and $\frac{15}{4}-y$.

Not all these amounts can be integers (for example, it is impossible for both $\frac{15}{4}+x$ and $\frac{15}{4}-x$ to be integers), so in particular, these amounts cannot be the original amounts of 1, 2, 4, and 8. This means that further blends are required to get to these amounts. However, a blend always produces two equal amounts. The only way that there are two equal amounts among $\frac{15}{4}+x$, $\frac{15}{4}-x$, $\frac{15}{4}+y$, and $\frac{15}{4}-y$ is if $x = y$.

Hence, our amounts are now $\frac{15}{4}+x$, $\frac{15}{4}+x$, $\frac{15}{4}-x$, and $\frac{15}{4}-x$. Since a blend always produces two equal amounts, we see that at some point, the father performed a blend to produce two equal amounts of $\frac{15}{4}+x$, and another blend to produce two equal amounts of $\frac{15}{4}-x$. We have shown that the following four blends are required:

(A) A blend to produce two equal amounts of $\frac{15}{4}+x$.

(B) A blend to produce two equal amounts of $\frac{15}{4}-x$.

(C) Blending an amount of $\frac{15}{4}+x$ and an amount of $\frac{15}{4}-x$ to produce two equal amounts of $\frac{15}{4}$.

(D) Blending another amount of $\frac{15}{4}+x$ and another amount of $\frac{15}{4}-x$ to produce two equal amounts of $\frac{15}{4}$.

The father performs a total of four blends, so these must be the only blends.

Blends (A) and (B) must come before blends (C) and (D). After blends (A) and (B), the amounts are $\frac{15}{4}+x$, $\frac{15}{4}+x$, $\frac{15}{4}-x$, and $\frac{15}{4}-x$. After any first blend, the amounts are always of the form a, a, b, and c, where a, b, and c are distinct. (We know that a, b, and c are distinct because among the original amounts of 1, 2, 4, and 8, no amount is equal to the average of any other two amounts.) The only way to get two pairs of equal amounts after the next blend is to blend the amounts of b and c. The probability that this occurs is $\frac{1}{\binom{4}{2}} = \frac{1}{6}$.

We then have the amounts $\frac{15}{4}+x$, $\frac{15}{4}+x$, $\frac{15}{4}-x$, and $\frac{15}{4}-x$. The next blend must blend an amount of $\frac{15}{4}+x$ and an amount of $\frac{15}{4}-x$. The probability that this occurs is $\frac{2\cdot 2}{\binom{4}{2}} = \frac{4}{6} = \frac{2}{3}$.

Finally, we have the amounts $\frac{15}{4}$, $\frac{15}{4}$, $\frac{15}{4}+x$, and $\frac{15}{4}-x$. The next blend must blend the amounts of $\frac{15}{4}+x$ and $\frac{15}{4}-x$. The probability that this occurs is $\frac{1}{\binom{4}{2}} = \frac{1}{6}$.

Therefore, the probability that after four blends, all four children have the same amount of mashed potatoes is

$$\frac{1}{6}\cdot\frac{2}{3}\cdot\frac{1}{6} = \frac{2}{108} = \frac{1}{54}.$$

3/2/23.

Find all integers b such that there exists a positive real number x with

$$\frac{1}{b} = \frac{1}{\lfloor 2x \rfloor} + \frac{1}{\lfloor 5x \rfloor}.$$

Here $\lfloor y \rfloor$ denotes the greatest integer that is less than or equal to y.

We claim that the only such positive integers b are 3 and all positive multiples of 10.

Let $n = \lfloor x \rfloor$. Then there exists a unique integer r, $0 \le r \le 9$, such that

$$n + \frac{r}{10} \le x < n + \frac{r+1}{10}.$$

For each such value of r, we can express $\lfloor 2x \rfloor$ and $\lfloor 5x \rfloor$ in terms of n.

r	$\lfloor 2x \rfloor$	$\lfloor 5x \rfloor$
0	$2n$	$5n$
1	$2n$	$5n$
2	$2n$	$5n+1$
3	$2n$	$5n+1$
4	$2n$	$5n+2$
5	$2n+1$	$5n+2$
6	$2n+1$	$5n+3$
7	$2n+1$	$5n+3$
8	$2n+1$	$5n+4$
9	$2n+1$	$5n+4$

Solving for b in the given equation, we find

$$b = \frac{\lfloor 2x \rfloor \lfloor 5x \rfloor}{\lfloor 2x \rfloor + \lfloor 5x \rfloor}.$$

We divide into cases.

Case 1: $r = 0$ or 1. In this case,

$$b = \frac{2n \cdot 5n}{2n + 5n} = \frac{10n}{7}.$$

If b is an integer, then n must be a multiple of 7. Let $n = 7k$. Then $b = 10k$, so b can take on all positive multiples of 10.

Case 2: $r = 2$ or 3. In this case,

$$b = \frac{2n(5n+1)}{2n+(5n+1)} = \frac{10n^2+2n}{7n+1}.$$

If b is an integer, then

$$7b - 10n = \frac{70n^2+14n}{7n+1} - 10n = \frac{4n}{7n+1}$$

must be an integer. But $0 < \frac{4n}{7n+1} < 1$, so b cannot be an integer.

Case 3: $r = 4$. In this case,

$$b = \frac{2n(5n+2)}{2n+(5n+2)} = \frac{10n^2+4n}{7n+2}.$$

If b is an integer, then

$$7b - (10n+1) = \frac{70n^2+28n}{7n+2} - (10n+1) = \frac{n-2}{7n+2}.$$

must be an integer. If $n = 2$, then $b = 3$.

If $n = 1$, then $b = 14/9$, which is not an integer, and if $n \geq 3$, then $0 < \frac{n-2}{7n+2} < 1$, so b is not an integer.

Case 4: $r = 5$. In this case,

$$b = \frac{(2n+1)(5n+2)}{(2n+1)+(5n+2)} = \frac{10n^2+9n+2}{7n+3}.$$

If b is an integer, then

$$7b - (10n+4) = \frac{70n^2+63n+14}{7n+3} - (10n+4) = \frac{5n+2}{7n+2}.$$

must be an integer. But $0 < \frac{5n+2}{7n+2} < 1$, so b cannot be an integer.

Case 5: $r = 6$ or 7. In this case,

$$b = \frac{(2n+1)(5n+3)}{(2n+1)+(5n+3)} = \frac{10n^2+11n+3}{7n+4}.$$

If b is an integer, then

$$7b - (10n+5) = \frac{70n^2+77n+21}{7n+4} - (10n+5) = \frac{2n+1}{7n+2}.$$

must be an integer. But $0 < \frac{2n+1}{7n+2} < 1$, so b cannot be an integer.

Case 6: $r = 8$ or 9. In this case,

$$b = \frac{(2n+1)(5n+4)}{(2n+1)+(5n+4)} = \frac{10n^2+13n+4}{7n+5}.$$

If b is an integer, then

$$7b - (10n+5) = \frac{70n^2+91n+28}{7n+5} - (10n+5) = \frac{6n+3}{7n+5}.$$

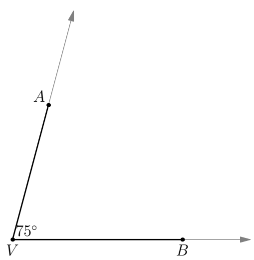
must be an integer. But $0 < \frac{6n+3}{7n+5} < 1$, so b cannot be an integer.

Therefore, the only possible values of b are 3 and all positive multiples of 10.

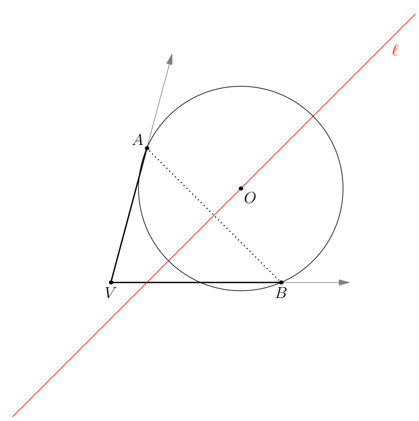
4/2/23.

A *luns* with vertices X and Y is a region bounded by two circular arcs meeting at the endpoints X and Y. Let A, B, and V be points such that $\angle AVB = 75°$, $AV = \sqrt{2}$ and $BV = \sqrt{3}$. Let L be the largest area luns with vertices A and B that does not intersect the lines $\overleftrightarrow{VA}$ or $\overleftrightarrow{VB}$ in any points other than A and B. Define k as the area of L. Find the value

$$\frac{k}{(1+\sqrt{3})^2}.$$

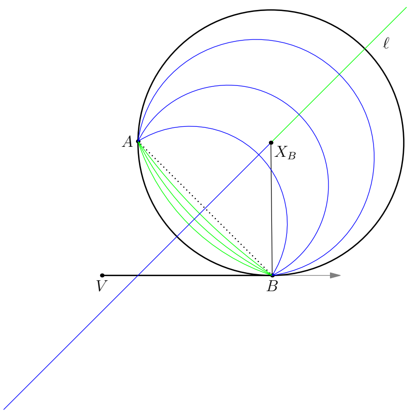


We first consider the question of which circular arcs from A to B lie entirely inside the region bounded by the $75°$ angle to the left. Any such arc is the arc of a circle with center equidistant from A and B. Therefore the locus of possible centers for these arcs lie on the perpendicular bisector, ℓ, of the segment AB.



If we choose a center for our circle, the circle defines two different arcs, one to the "left" of AB and one to the "right" or AB. For the sake of this problem, we will define **left** as pertaining to the half-plane bounded by $\overleftrightarrow{AB}$ containing V and **right** as pertaining to the half-plane bounded by $\overleftrightarrow{AB}$ not containing V.

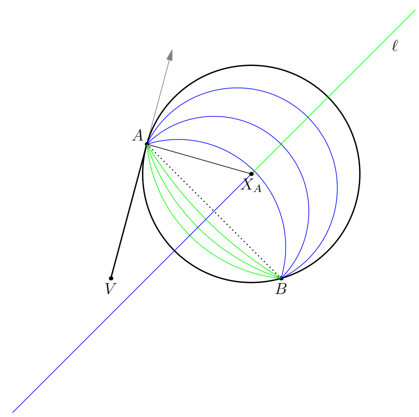
For any given center, we must explore whether either or both of these arcs intersects either the line $\overleftrightarrow{VA}$ or the line $\overleftrightarrow{VB}$. Let O be the center of an arbitrary circle that intersects $\overleftrightarrow{VB}$ at B. (Note that O and B uniquely determine the circle.) If this circle is tangent to $\overleftrightarrow{VB}$, then $\angle VBO = 90°$. If the circle intersects $\overleftrightarrow{VB}$ to the left of B then $\angle VBO < 90°$. If the circle intersects $\overleftrightarrow{VB}$ to the right of B then $\angle VBO > 90°$.

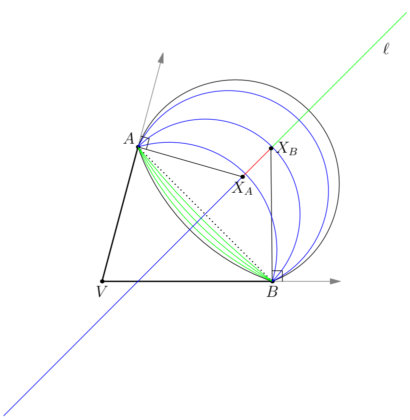


In the figure to the left, we've drawn the lines $\overleftrightarrow{VB}$ and ℓ. The point X_B is the intersection of ℓ and the perpendicular to $\overleftrightarrow{VB}$ at B. The point X_B is the center of the black circle and the black circle intersects $\overleftrightarrow{VB}$ only at B. In particular, both arcs from A to B in this circle lie above the line $\overleftrightarrow{VB}$, (excepting the point B).

The green region of ℓ is the set of points on ℓ to the right of $\overleftrightarrow{X_BB}$. These are the centers of circles that intersect $\overleftrightarrow{VB}$ to the right of B. For such a circle, only the left arc from A to B lies above $\overleftrightarrow{VB}$. The blue region of ℓ is the set of points on ℓ to the left of $\overleftrightarrow{X_BB}$. These are the centers of the circles that intersect $\overleftrightarrow{VB}$ to the left of B. For such a circle, only the right arc from A to B lies above $\overleftrightarrow{VB}$. Notice that every valid arc lies inside the circle centered at X_B containing the point B.

We consider an identical construction for the line $\overleftrightarrow{VA}$ in the figure to the right. The point X_A is the center of the unique circle for which both the left and right arcs from A to B do not intersect $\overleftrightarrow{VA}$. The green points to the right of X_A are the centers of the circles for which the left arc from A to B does not intersect $\overleftrightarrow{VA}$. The blue points to the left of X_A are those points for which the right arc does not intersect $\overleftrightarrow{VA}$. Notice that all of these arcs are in the interior of the circle centered at X_A.





Next we combine these figures. We drop both perpendiculars through A and B. Since $VB > VA$, the point X_B is to the right of X_A. If the center of a circle lies on $\overline{X_AX_B}$, then the left arc of the circle intersects $\overleftrightarrow{VB}$ twice and the right arc of the circle intersects $\overleftrightarrow{VA}$ twice. Therefore neither arc is an arc of a luns.

If the center of a circle lies to the right of X_B on ℓ (colored green here), then the left arc of this circle does not intersect either line except at A and B. If the center of a circle lies to the left of X_A (colored blue), then the right arc of the circle does not intersect either line, except at A and B. Therefore the green region of ℓ parameterizes the set of all valid left arcs and the blue region of ℓ parameterizes all of the valid right arcs.

Consider the black luns in this figure. It has left arc with center X_B and right arc with center X_A. This figure is a luns, and every valid luns is bounded by a pair of arcs that lie inside this figure. Therefore every valid luns is a subset of this luns and this luns has the maximal area of any luns satisfying the assumptions. Now we compute this area, by computing the sum of the areas of the green region and the blue region.

Define the lengths $a = VB = \sqrt{3}$, $b = VA = \sqrt{2}$, and $c = AB$. The law of cosines gives

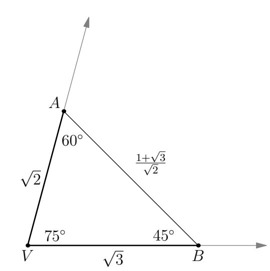
$$\begin{aligned} c^2 &= (VA)^2 + (VB)^2 - 2(VA)(VB)\cos 75^\circ \\ &= 2 + 3 - 2\sqrt{2}\cdot\sqrt{3}\cdot\frac{\sqrt{6}-\sqrt{2}}{4} \\ &= 5 - 2\sqrt{6}\cdot\frac{\sqrt{6}-\sqrt{2}}{4} \\ &= 5 - \frac{6-2\sqrt{3}}{2} \\ &= 2+\sqrt{3}. \end{aligned}$$

Notice also that, since

$$(1+\sqrt{3})^2 = 4 + 2\sqrt{3} = 2c^2,$$

and $1+\sqrt{3}$ is positive,

$$c = \frac{1+\sqrt{3}}{\sqrt{2}}.$$



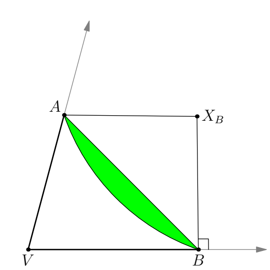
Finally, we apply the law of sines to find the other angles in VAB. Since

$$\frac{c}{\sin\angle V} = \frac{\frac{1+\sqrt{3}}{\sqrt{2}}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{4(1+\sqrt{3})}{2\sqrt{3}+2} = 2,$$

we know that

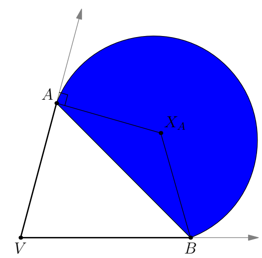
$$2 = \frac{b}{\sin\angle B} = \frac{\sqrt{2}}{\sin\angle B},$$

so $\sin\angle B = \frac{1}{\sqrt{2}}$ and $\angle VBA = 45^\circ$. Subtracting gives $\angle VAB = 60^\circ$.



First we compute the area of the green region. Since $\angle ABV = 45^\circ$ and $AX_B = BX_B$, the triangle AX_BB is right isosceles. The sector containing the green region is a quarter of a circle of radius $\frac{c}{\sqrt{2}}$, so the entire sector has area $\frac{1}{4}\cdot\pi\cdot\left(\frac{c}{\sqrt{2}}\right)^2 = \frac{\pi c^2}{8}$. To find the green region we subtract the area of the triangle to get

$$\frac{\pi c^2}{8} - \frac{c^2}{4} = \frac{\pi - 2}{8}\cdot c^2.$$



Next we compute the area of the blue region. Since triangle AX_AB is isosceles and $\angle VAB = 60^\circ$, we get that $\angle X_A = 120^\circ$. Therefore the blue region is the union of $\frac{2}{3}$ of the circle with center X_A plus the area of triangle AX_AB. The triangle has altitude $\frac{c}{2\sqrt{3}}$ with respect to base AB and the radius of this circle is $\frac{c}{\sqrt{3}}$. Therefore the total area of the blue region is

$$\frac{2}{3} \cdot \pi \cdot \left(\frac{c}{\sqrt{3}}\right)^2 + \frac{1}{2} \cdot c \cdot \frac{c}{2\sqrt{3}} = \frac{3\sqrt{3} + 8\pi}{36} \cdot c^2.$$

This makes the total area of the luns

$$k = \frac{\pi - 2}{8} \cdot c^2 + \frac{3\sqrt{3} + 8\pi}{36} \cdot c^2 = \frac{6\sqrt{3} + 25\pi - 18}{72} \cdot c^2.$$

Since $c^2 = \frac{(1+\sqrt{3})^2}{2}$,

$$\frac{k}{(1+\sqrt{3})^2} = \frac{k}{2c^2} = \frac{6\sqrt{3} + 25\pi - 18}{144}.$$

5/2/23.

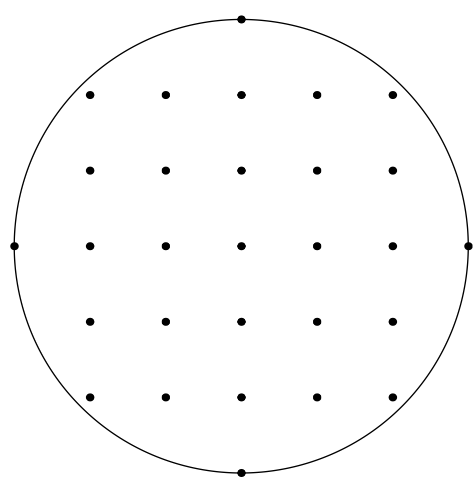
Miss Levans has 169 students in her history class and wants to seat them all in a 13×13 grid of desks. Each desk is placed at a different vertex of a 12 meter by 12 meter square grid of points she has marked on the floor. The distance between neighboring vertices is exactly 1 meter.

Each student has at most three best friends in the class. Best-friendship is mutual: if Lisa is one of Shannon's best friends, then Shannon is also one of Lisa's best friends. Miss Levans knows that if any two best friends sit at points that are 3 meters or less from each other then they will be disruptive and nobody will learn any history. And that is bad.

Prove that Miss Levans can indeed place all 169 students in her class without any such disruptive pairs.

There are only a finite number of ways of arranging the 169 students among the 169 desks. Consider the arrangement that has the least number of disruptive pairs. We claim that this minimal arrangement has no disruptive pairs.

For the sake of contradiction, suppose that there is a disruptive pair in this arrangement. We say that two students are *neighbors* if they sit within 3 meters of each other. As shown below, every student has at most 28 neighbors.



Let students S and T be a disruptive pair, which means they are best friends and neighbors. Let A be the set of all best friends of neighbors of S, so $|A| \leq 3 \cdot 28 = 84$. Let B be the set of all neighbors of best friends of S, so $|B| \leq 28 \cdot 3 = 84$. Note that S is in both A and B, so there are at most $83 + 83 = 166$ members in $A \cup B$ other than S. But there are a total of 169 students, so there must be at least $169 - 1 - 166 = 2$ students, other than S, who are not in A or B.

Let S' be one of these students, other than T. We swap the locations of students S and S'. This breaks up the disruptive pair S and T. Since S' is not in A, S' is not best friends with any of his neighbors in his new location (where S originally was), and since S' is not in B, S is not best friends with any of his neighbors in his new location (where S' originally was) either. So by swapping S and S', we have reduced the number of disruptive pairs by at least one. We now have an arrangement with less disruptive pairs, which contradicts the minimality assumption. Therefore, in the minimal arrangement, there are no disruptive pairs, so such a seating is possible.

Credits: Problem 4/2/23 is based on a proposal by Sandor Lehoczyk.
Problem 5/2/23 is based on a problem from the Bulgarian journal Matematika.
All other problems and solutions by USAMTS staff.