USA Mathematical Talent Search

Round 1 Solutions

Year 23 — Academic Year 2011–2012

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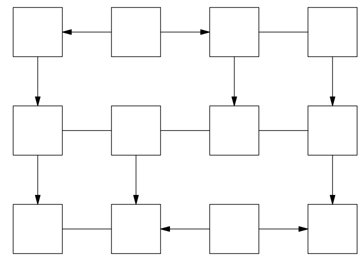
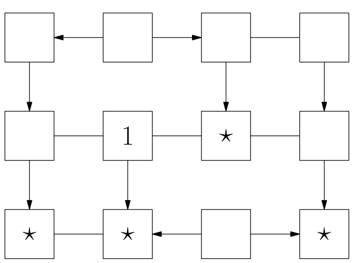


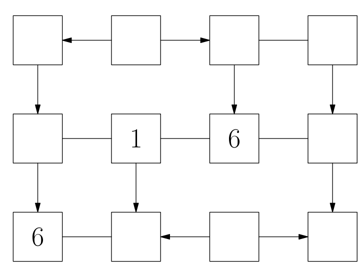
1/1/23. The grid on the right has 12 boxes and 15 edges connecting boxes. In each box, place one of the six integers from 1 to 6 such that the following conditions hold:

- For each possible pair of distinct numbers from 1 to 6, there is exactly one edge connecting two boxes with that pair of numbers.
- If an edge has an arrow, then it points from a box with a smaller number to a box with a larger number.

You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

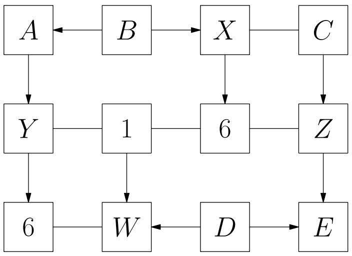


First, we notice that each number 1 through 6 must be placed in exactly two boxes. For each number, one of the two boxes must have 2 neighbors and the other must have 3. Of the boxes with 3 neighbors, there is exactly one that does not have any incoming arrows. This box must contain the value 1. Likewise, only two of the boxes with three neighbors have no outgoing arrows, and two of the boxes with two neighbors have no outgoing arrows. Therefore the two sixes must lie in these four boxes. We label these as $\star$ in our first diagram.

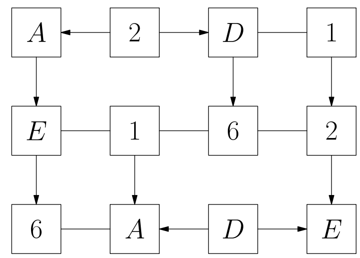


We note that wherever the two sixes land, they must have 5 distinct neighbors. In particular, they cannot land in a pair of boxes that share a neighbor. Among these four, the only 2- and 3-valent boxes (boxes with 2 or 3 neighbors respectively) that do not share a neighbor are the two to the right.

Now that we've taken care of the simplest cases, we will need to be much more careful with the remaining boxes. Let's label them as in the first figure on the next page, using letters early in the alphabet to represent the 2-valent boxes (containing 1-5) and letters late in the alphabet for the 3-valent boxes (containing 2-5).



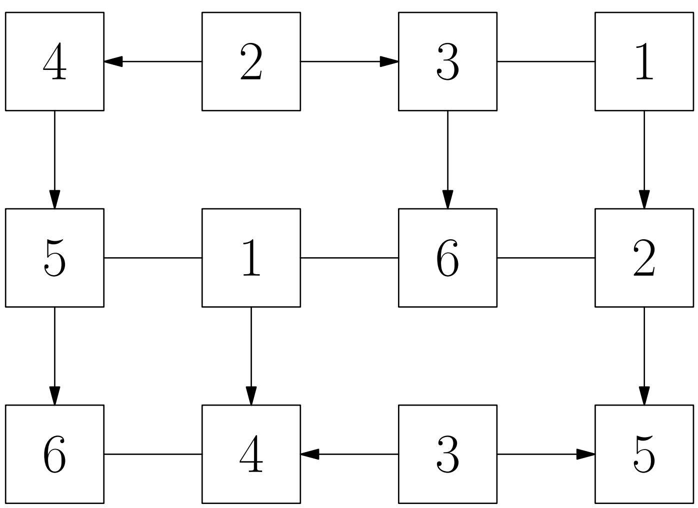
The values of D and E must be equal to two of the values from X, Y, Z, and W. However, we know that two boxes with the same value cannot be neighbors and also cannot share neighbors, so the values D and E must equal the values X and Y in some order. By the symmetric argument, the values A and B must be equal to the values Z and W in some order. The specific upshot of this is that we now know that we must have $C = 1$.



The remaining entries are all from 2 to 5. We know that A, X, and Y are distinct and all greater than B, so $B = 2$. Recall also that B must be equal to one of Z or W. Since $W > D$, W cannot be equal to 2, and so Z must be 2. Since $\{A, B\} = \{W, Z\}$ and $B = Z$, we must have $W = A$. Along the same lines, since $W = A$ and A neighbors Y, we cannot have $Y = D$, so instead we must have $Y = E$ and $X = D$. We compile this to the right.

Finishing the grid is straightforward from here. We know that A, D, and E must be equal to 3, 4, and 5 in some order. The bottom of the grid tells us that D is the smallest of the three, so $D = 3$. The top of the grid tells us that $A < E$, so we have $A = 4$ and $E = 5$.

The solution is given below, and a direct check shows that all conditions hold. Our argument above shows that this solution is unique.



2/1/23. Find all integers a, b, c, d, and e, such that

$$\begin{aligned} a^2 &= a + b - 2c + 2d + e - 8, \\ b^2 &= -a - 2b - c + 2d + 2e - 6, \\ c^2 &= 3a + 2b + c + 2d + 2e - 31, \\ d^2 &= 2a + b + c + 2d + 2e - 2, \\ e^2 &= a + 2b + 3c + 2d + e - 8. \end{aligned}$$

We begin by bringing all terms to the left sides,

$$\begin{aligned} a^2 - a - b + 2c - 2d - e + 8 &= 0, \\ b^2 + a + 2b + c - 2d - 2e + 6 &= 0, \\ c^2 - 3a - 2b - c - 2d - 2e + 31 &= 0, \\ d^2 - 2a - b - c - 2d - 2e + 2 &= 0, \\ e^2 - a - 2b - 3c - 2d - e + 8 &= 0. \end{aligned}$$

When we add these five equations together we get

$$a^2 - 6a + b^2 - 4b + c^2 - 2c + d^2 - 10d + e^2 - 8e + 55 = 0.$$

Now we can complete the squares,

$$(a^2 - 6a + 9) + (b^2 - 4b + 4) + (c^2 - 2c + 1) + (d^2 - 10d + 25) + (e^2 - 8e + 16) = 0,$$

or

$$(a - 3)^2 + (b - 2)^2 + (c - 1)^2 + (d - 5)^2 + (e - 4)^2 = 0.$$

From here we see that any quintuple (a, b, c, d, e) satisfying the original equations must necessarily satisfy this final equation. However, since squares of integers are always nonnegative, this can only be solved when each of the squares is equal to zero. Therefore the only possible solution is

$$(a, b, c, d, e) = (3, 2, 1, 5, 4).$$

We now need to check that these values satisfy all of the original equations.

To check that these values give a solution we must substitute them all back into the original equations. Indeed, all 5 equations are satisfied by these values,

$$\begin{aligned} 3^2 &= 3 + 2 - 2 \cdot 1 + 2 \cdot 5 + 4 - 8, \\ 2^2 &= -3 - 2 \cdot 2 - 1 + 2 \cdot 5 + 2 \cdot 4 - 6, \\ 1^2 &= 3 \cdot 3 + 2 \cdot 2 + 1 + 2 \cdot 5 + 2 \cdot 4 - 31, \\ 5^2 &= 2 \cdot 3 + 2 + 1 + 2 \cdot 5 + 2 \cdot 4 - 2, \\ 4^2 &= 3 + 2 \cdot 2 + 3 \cdot 1 + 2 \cdot 5 + 4 - 8. \end{aligned}$$

Therefore there is a single, unique answer: $(a, b, c, d, e) = (3, 2, 1, 5, 4)$.

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3/1/23. **(Corrected from an earlier release.)** You have 14 coins, dated 1901 through 1914. Seven of these coins are real and weigh 1.000 ounce each. The other seven are counterfeit and weigh 0.999 ounces each. You do not know which coins are real or counterfeit. You also cannot tell which coins are real by look or feel.

Fortunately for you, Zoltar the Fortune-Weighing Robot is capable of making very precise measurements. You may place any number of coins in each of Zoltar's two hands and Zoltar will do the following:

- If the weights in each hand are equal, Zoltar tells you so and returns all of the coins.
- If the weight in one hand is heavier than the weight in the other, then Zoltar takes one coin, at random, from the heavier hand as tribute. Then Zoltar tells you which hand was heavier, and returns the remaining coins to you.[1]

Your objective is to identify a single real coin that Zoltar has not taken as tribute. Is there a strategy that guarantees this? If so, then describe the strategy and why it works. If not, then prove that no such strategy exists.

We begin the process by placing one coin in each of Zoltar's hands. If the coins are the same weight, we replace the coin in Zoltar's left hand with a new coin and eventually the hands will be imbalanced. Zoltar then identifies a real and a fake coin and takes the real coin as tribute. We now have 13 coins left: 6 of these are real and we have 1 known counterfeit coin.

If we set the known counterfeit coin aside, we reduce the problem to the situation where we now have 6 real coins and 6 counterfeit coins. We again weigh some chosen coin against the other 11 coins until Zoltar takes a real coin and leaves a known counterfeit coin. Discarding the counterfeit again lets us reduces the problem and we now have 5 real and 5 counterfeit coins.

If we repeat this reduction process three more times we will reduce our set of coins to exactly 4 unknown coins, 2 of which are real and 2 of which are counterfeit. Now we place two coins in each of Zoltar's hands. One of the three possible pairings that we can try will lead to an imbalance. This imbalance happens when both real coins are in one of his hands and both counterfeit coins are in the other. When this happens, Zoltar will take one of the real coins and we are assued that the remaining coin from that side of the scale is also real.

[1] In the earlier version, this sentence read, "Then Zoltar tells you *the result of the measurement*, and returns the remaining coins to you." The correction clarifies that "the result of the measurement" was meant to refer only to which hand was heavier, not to the actual weight in either hand.

This process requires exactly 6 unbalanced weighings and leaves us with 8 coins. Seven of these coins are known counterfeits and the eighth is the desired real coin.

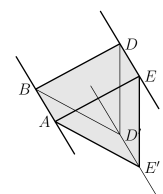
4/1/23. Let $ABCDEF$ and $ABC'D'E'F'$ be regular planar hexagons in three-dimensional space with side length 1, such that $\angle EAE' = 60°$. Let $\mathcal{P}$ be the convex polyhedron whose vertices are A, B, C, C', D, D', E, E', F, and F'.

(a) Find the radius r of the largest sphere that can be enclosed in polyhedron $\mathcal{P}$.

(b) Let $\mathcal{S}$ be a sphere enclosed in polyhedron $\mathcal{P}$ with radius r (as derived in part (a)). The set of possible centers of $\mathcal{S}$ is a line segment $\overline{XY}$. Find the length XY.

(a) This polyhedron consists of 2 hexagonal faces meeting along an edge, a pair of opposite triangles, $\triangle AFF'$ and $\triangle BCC'$, and three quadrilaterals connecting the remaining pairs of corresponding sides. We are looking for the set of all spheres of maximal radius that fit inside the region bounded by the 6 planes containing these polygons.

We begin by studying only the three planes containing the two hexagons and the quadrilateral $DEE'D'$. We will find the maximal radius for a sphere bounded by these planes, giving an upper bound for the solution of the problem, and then show that such a sphere can be placed inside our polyhedron.

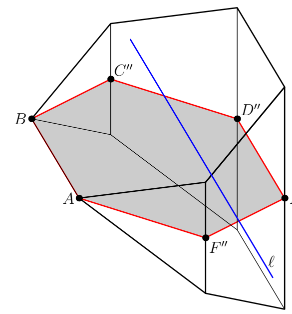


Consider the triangular prism $\overline{\mathcal{T}}$ bounded by the three segments $\overline{AB}$, $\overline{ED}$, and $\overline{E'D'}$, and let $\mathcal{T}$ be the infinite triangular prism bounded by the lines containing these segments. These three lines are parallel and we take a cross-section of $\mathcal{T}$ orthogonal to these lines. Since $\angle EAE' = 60°$, this cross-section is an equilateral triangle, and the radius of the largest sphere that can be enclosed in $\mathcal{T}$ is the inradius of this triangle.

Since triangle ADE is a 30°-60°-90° triangle and $DE = 1$, the side length of the equilateral triangle is $AE = \sqrt{3}$. The inradius of an equilateral triangle with side length s is $\frac{s}{2\sqrt{3}}$, so the inradius of this triangle is $\frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2}$. Therefore the maximal radius of any sphere bounded by $\mathcal{T}$ is $\frac{1}{2}$. This tells us we have an upper bound of $\frac{1}{2}$ on the radius of any sphere living inside $\mathcal{P}$.

On the other hand, notice that since $AB = 1$, there does exist a sphere of radius $\frac{1}{2}$ (so diameter 1) tangent to all of the faces of $\overline{\mathcal{T}}$. Since $\overline{\mathcal{T}}$ lies inside $\mathcal{P}$ this shows that $\frac{1}{2}$ is achievable as the maximal radius of sphere enclosed by $\mathcal{P}$.

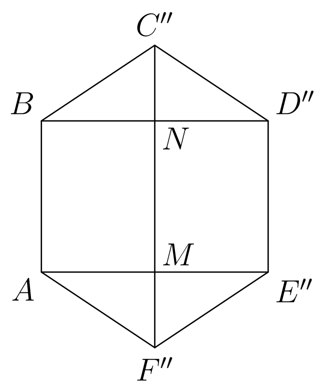
(b) We learn slightly more from the previous part. We know that all spheres enclosed by $\mathcal{P}$ of radius $\frac{1}{2}$ must be tangent to the faces of $\mathcal{T}$, so the centers of these spheres must lie on its line of symmetry: the line ℓ containing the incenters of $\triangle AEE'$ and $\triangle BDD'$. The boundary of the desired segment $\overline{XY}$ is determined by the centers of the spheres that are tangent to the remaining four faces of $\mathcal{P}$. We find these boundary points now.



Let C'', D'', E'', and F'' be the midpoints of $\overline{CC'}$, $\overline{DD'}$, $\overline{EE'}$, and $\overline{FF'}$, respectively. By symmetry, the hexagon $ABC''D''E''F''$ is planar and lies on the plane containing $\overline{AB}$ and ℓ.

Since the lines $\overleftrightarrow{CC'}$, $\overleftrightarrow{DD'}$, $\overleftrightarrow{EE'}$, and $\overleftrightarrow{FF'}$ are all orthogonal to the plane of $ABC''D''E''F''$, the five faces of P that we have not yet discussed are also all orthogonal to this plane. Therefore any sphere with center in the plane of $ABC''D''E''F''$ that is tangent to any of the other faces will be tangent at a point on the hexagon $ABC''D''E''F''$. This reduces the problem to classifying all circles with radius $\frac{1}{2}$ that are enclosed by this hexagon. We now solve this reduced problem.

Let M and N be the midpoints of $\overline{AE''}$ and $\overline{BD''}$, respectively. Then $\overline{MN}$ is parallel to $\overline{AB}$, and $MN = AB = 1$. By symmetry, M and N lie on $\overline{C''F''}$, and $C''N = F''M$, so $C''N = F''M = (C''F'' - MN)/2 = (2-1)/2 = 1/2$.



Since BDD' is an equilateral triangle, triangle BDD'' is a 30°-60°-90° triangle. Then

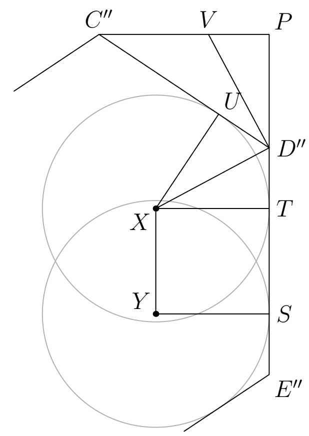
$$BD'' = \frac{\sqrt{3}}{2}BD = \frac{\sqrt{3}}{2}\cdot\sqrt{3} = \frac{3}{2}.$$

Hence, $BN = ND'' = BD''/2 = 3/4$. Then by Pythagoras on rigtht triangle $C''ND''$,

$$\begin{aligned} C''D'' &= \sqrt{(C''N)^2 + (ND'')^2} \\ &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2} \\ &= \sqrt{\frac{1}{4} + \frac{9}{16}} \\ &= \sqrt{\frac{13}{16}} \\ &= \frac{\sqrt{13}}{4}. \end{aligned}$$

The line ℓ is located a distance $\frac{1}{2}$ from the D'' and E'', so specifically is closer to $\overline{D''E''}$ than to $\overline{AB}$. Since $\overleftrightarrow{C''F''}$ is a line of symmetry of this hexagon the extremal enclosed circles must be tangent to $\overleftrightarrow{C''D''}$ or $\overleftrightarrow{E''F''}$. One endpoint of line segment $\overline{XY}$, say X, is at a distance of 1/2 from the segment $\overline{C''D''}$, and the other endpoint Y is at a distance of 1/2 from the segment $\overline{E''F''}$.

Let the sphere centered at X with radius $1/2$ be tangent to $\overline{D''E''}$ and $\overline{C''D''}$ at T and U, respectively. Let P be the projection of C'' onto $\overline{D''E''}$, and let the angle bisector of $\angle PD''C''$ intersect $\overline{PC''}$ at V. Then $\overline{D''V}$ and $\overline{D''X}$ are the internal and external angle bisectors of $\angle PD''C''$, respectively, so they are perpendicular. This implies that right triangles $D''PV$ and XTD'' are similar. Since $XT = PD'' = 1/2$, the two triangles are in fact congruent.



By the angle bisector theorem,

$$PV = \frac{PD''}{PD'' + C''D''} \cdot PC'' = \frac{1/2}{1/2 + \sqrt{13}/4} \cdot \frac{3}{4} = \frac{3}{2(2+\sqrt{13})}.$$

Rationalizing the denominator, we get

$$PV = \frac{3(\sqrt{13}-2)}{2(\sqrt{13}+2)(\sqrt{13}-2)} = \frac{3(\sqrt{13}-2)}{2 \cdot 9} = \frac{\sqrt{13}-2}{6}.$$

Hence,

$$TD'' = PV = \frac{\sqrt{13}-2}{6}.$$

Let the sphere centered at Y with radius $1/2$ be tangent to $\overline{D''E''}$ at S. Then by symmetry, $SE'' = TD''$, so

$$XY = ST = D''E'' - TD'' - SE'' = 1 - 2 \cdot \frac{\sqrt{13}-2}{6} = \frac{5-\sqrt{13}}{3}.$$

5/1/23. In the game of Tristack Solitaire, you start with three stacks of cards, each with a different positive integer number of cards. At any time, you can double the number of cards in any one stack of cards by moving cards from exactly one other, larger, stack of cards to the stack you double. You win the game when any two of the three stacks have the same number of cards.

For example, if you start with stacks of 3, 5, and 7 cards, then you have three possible legal moves:

- You may move 3 cards from the 5-card stack to the 3-card stack, leaving stacks of 6, 2, and 7 cards.
- You may move 3 cards from the 7-card stack to the 3-card stack, leaving stacks of 6, 5, and 4 cards.
- You may move 5 cards from the 7-card stack to the 5-card stack, leaving stacks of 3, 10, and 2 cards.

Can you win Tristack Solitaire from any starting position? If so, then give a strategy for winning. If not, then explain why.

We present two solutions to this problem.

Solution 1:

First, we show the winning strategy for a particular starting configuration:

Lemma 1: Suppose the three stacks have a, b, and c cards, where $0 < a \leq b < c$, and suppose that b is a multiple of a; that is, $b = na$ for some positive integer n. Then there is a winning strategy in which the first two stacks (the stacks that start with a and b cards) result in the same number of cards.

Proof of Lemma 1: For ease of referral, call the stack that starts with a cards the "first" stack, the stack that starts with b cards the "middle" stack, and the stack that starts with c cards the "last" stack. Note that if $n = 1$ then we've already won, so assume that $n > 1$.

Write $n = (1d_{k-1} \ldots d_1 d_0)_2$ as a base-2 integer, where $k = \lfloor \log_2 n \rfloor$ and each d_i is a binary digit (0 or 1). Then we can make the first and middle stacks equal in k steps, as follows:

> On step i (for $0 \leq i < k$), if $d_i = 1$, then double the first stack by moving cards from the middle stack, and if $d_i = 0$, then double the first stack by moving cards from the last stack.

At the end of the process, we will have doubled the first stack k times, so there will be $2^k a$ cards in the first stack. Since we are removing 2^i cards from the middle stack if and only if $d_i = 1$ for all $0 \le i < k$, we will remove exactly $(d_{k-1}d_{k-2}\ldots d_1 d_0)_2 a$ cards from the middle stack, leaving $(1000\ldots 0)_2 a = 2^k a$ cards in the middle stack. Thus the first and middle stacks will be equal. Since we are moving a total of $(2^k - 1)a$ cards, and $(2^k - 1)a < 2^k a \le b < c$, we will always have sufficient cards in the last stack for all the steps in which $d_i = 0$. $\square$

Lemma 1 establishes that we can win the game if the middle stack is an integer multiple of the smallest stack. Next, we show that if condition does not hold, then we can "improve" our position in a well-defined way.

Lemma 2: Suppose the three stacks have a, b, and c cards, where $0 < a < b < c$, and suppose that b is not a multiple of a. Then there is a series of moves that produces of stack of size r, where $1 \le r < a$.

Proof of Lemma 2: Write $b = na + r$, where n is a positive integer and $0 < r < a$. (Note that r is simply the remainder upon division of b by a). We perform exactly the same algorithm as in Lemma 1. After this algorithm concludes, we have stacks of size $2^k a$, $2^k a + r$, and a third stack whose size is irrelevant to this argument. One additional move of doubling the $2^k a$ stack from the $2^k a + r$ stack leaves us with our stack of size r. $\square$

Proof of the Original Claim: Therefore, we can either win right away (using Lemma 1), or we can use Lemma 2 to produce a position with a stack that is smaller than any of the stacks we started with. We repeat this use of Lemma 2 until we arrive at a position in which we can use Lemma 1 to win the game—we are guaranteed that this must happen within a finite number of uses of Lemma 2 (the "worst case" scenario is that the smallest stack only decreases by 1 upon each application of Lemma 2, but eventually it will decrease down to 1, at which point Lemma 1 will apply since the size of the medium stack will always be a multiple of 1).

Solution 2:

For an integer $n \ne 0$, we define $\operatorname{ord}_2(n)$ as the largest integer k such that 2^k divides n. For example, $\operatorname{ord}_2(24) = 3$.

Lemma 1: Given stacks initially containing a and b cards, where $\operatorname{ord}_2(a) > \operatorname{ord}_2(b)$, we can make a series of Tristack Solitaire moves on those two stacks that changes their respective sizes to a' and b', where $a' = a/2$ and $\min\{\operatorname{ord}_2(a'), \operatorname{ord}_2(b')\} = \min\{\operatorname{ord}_2(a), \operatorname{ord}_2(b)\}$.

Proof of Lemma 1: The sizes a and b are respectively even and odd multiples of $2^{\operatorname{ord}_2(b)}$, so $a + b$ is an odd multiple of $2^{\operatorname{ord}_2(b)}$.

Since $a \ne b$, we can make a unique Tristack Solitaire move on the two stacks. The number of cards moved is a multiple of $2^{\operatorname{ord}_2(b)}$, so both stack sizes remain multiples of $2^{\operatorname{ord}_2(b)}$ after the move. Moreover, one of the stack sizes must still be an even multiple of $2^{\operatorname{ord}_2(b)}$ and one must be an odd multiple of $2^{\operatorname{ord}_2(b)}$, since the sum of their sizes is still $a + b$. Therefore, the sizes of the two stacks remain unequal, and we can continue making moves on these two stacks perpetually. We also see that the minimum value of ord_2 over the sizes of the two stacks is invariant throughout this process.

Each move doubles the sizes of **both** stacks modulo $a+b$. Consider the set of multiples of $2^{\text{ord}_2(b)}$ modulo $a+b$. Doubling (mod $a+b$) is a one-to-one operation on this set, because $a+b$ is an odd multiple of $2^{\text{ord}_2(b)}$. In particular, doubling has an inverse operation on this set, so as we perform Tristack Solitaire moves on the two stacks, their sizes are periodic (mod $a+b$). This implies that their sizes are in fact periodic, since their sizes are always between 0 and $a+b$. Therefore, the stacks eventually return to their original sizes. Suppose this happens after k Tristack Solitaire moves ($k>0$). In that case, one stack must have move to half its original size after $k-1$ moves, since every Tristack Solitaire move doubles one stack. Moreover, this stack must be the first stack, because the minimum value of ord_2 over the two stack sizes cannot decrease. This proves the lemma. $\square$

Given two stacks with distinct ord_2, we refer to a series of Tristack Solitaire moves on those two stacks that halves the larger order stack as the **inverse move**.

Here is an algorithm for playing Tristack Solitaire:

1. If two stacks are equal, stop and declare victory. Otherwise, go to step 2.
2. If possible, choose two stacks of sizes a and b such that $\text{ord}_2(a)=\text{ord}_2(b)$. Perform the unique Tristack Solitaire move on those two stacks, then go to step 1. If there are no such stacks, go to step 3.
3. Choose two stacks of sizes a and b such that $\text{ord}_2(a)\geq\text{ord}_2(b)+2$. Perform an inverse move on those two stacks, then go to step 1.

Lemma 2: The algorithm can be performed.

Proof of Lemma 2: The only thing to check is that, when we get to step 3, we can always choose two stacks of sizes a and b such that $\text{ord}_2(a)\geq\text{ord}_2(b)+2$. This is true because we only get to step 3 from step 2, and then only if no two stacks have sizes of the same ord_2. In this case, we may denote the sizes of the stacks by a, b, and c, where $\text{ord}_2(a)>\text{ord}_2(c)>\text{ord}_2(b)$. It follows that $\text{ord}_2(a)\geq\text{ord}_2(b)+2$. $\square$

Lemma 3: Let ℓ and m respectively denote the largest and second largest of the values $\text{ord}_2(x),\text{ord}_2(y),\text{ord}_2(z)$, where x, y, and z are the sizes of three stacks at any given time. Each time we perform step 2 of our algorithm, m increases. Each time we perform step 3, m does not change, and $\ell-m$ decreases.

Proof of Lemma 3: When we perform step 2, we choose two stacks of sizes a and b such that $\text{ord}_2(a)=\text{ord}_2(b)$. Thus, a and b are both odd multiples of $2^{\text{ord}_2(b)}$. The number of cards we move is one of these odd multiples, so after the move, both stack sizes are even multiples of $2^{\text{ord}_2(b)}$. This means the ord_2 of both stack sizes increases, while the third stack is unchanged. The other stack had either the largest or smallest order (including the case that all orders are equal), so the value of m increases.

Now let us examine step 3. Suppose the stack sizes are a, b, and c, where $\operatorname{ord}_2(a) > \operatorname{ord}_2(c) > \operatorname{ord}_2(b)$. We perform the inverse move on the stacks of size a and b, changing their sizes to $a' = a/2$ and b'. Thus $\operatorname{ord}_2(a') = \operatorname{ord}_2(a) - 1 \geq \operatorname{ord}_2(c)$. We know that $\operatorname{ord}_2(b) < \operatorname{ord}_2(a) - 1$. It follows by Lemma 1 that $\operatorname{ord}_2(b') = \operatorname{ord}_2(b) < \operatorname{ord}_2(c)$. Therefore, m does not change, and $\ell - m$ decreases by 1. $\square$

Lemma 4: The algorithm terminates, and we win.

Proof of Lemma 4: Each time we perform step 3 of the algorithm, $\ell - m$ (as defined in Lemma 3) decreases, and we check the condition on step 2 before returning to step 3. Therefore, after finitely many steps, we have $\ell = m$, triggering a move in step 2. Each time this happens, m increases. There is an upper bound on m, since the sum of the stack sizes is constant. Therefore, the algorithm must terminate, and the only way this can happen is in step 1, where we win. $\blacksquare$

Credits: Problem 4/1/23 is based on a proposal by Luyi Zhang.
All other problems and solutions by USAMTS staff.