USA Mathematical Talent Search
Round 2 Solutions
Year 21 — Academic Year 2009–2010
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1/2/21. Jeremy has a magic scale, each side of which holds a positive integer. He plays the following game: each turn, he chooses a positive integer n. He then adds n to the number on the left side of the scale, and multiplies by n the number on the right side of the scale. (For example, if the turn starts with 4 on the left and 6 on the right, and Jeremy chooses $n = 3$, then the turn ends with 7 on the left and 18 on the right.) Jeremy wins if he can make both sides of the scale equal.

(a) Show that if the game starts with the left scale holding 17 and the right scale holding 5, then Jeremy can win the game in 4 or fewer turns.

(b) Prove that if the game starts with the right scale holding b, where $b \geq 2$, then Jeremy can win the game in $b - 1$ or fewer turns.

(a) Jeremy wins as follows:

Turn	n	Left	Right
Start		17	5
1	1	18	5
2	1	19	5
3	1	20	5
4	5	25	25

(b) Let a be the starting number on the left scale. First, notice that if on any turn Jeremy chooses $n = 1$, then the left side increases by 1 and the right side remains unchanged. Thus, our strategy is to have Jeremy choose 1 each turn until the left side is a multiple of $b - 1$. This will take at most $b - 2$ turns: if a is already a multiple of $b - 1$, no turns are needed; otherwise, if $a = d(b - 1) + r$, where $1 \leq r \leq b - 2$, the left side will be a multiple of $b - 1$ after $b - 1 - r \leq b - 2$ turns in which Jeremy chooses 1.

After this, the left side equals $k(b - 1)$, where k is some positive integer, and the right side is still b. If Jeremy now chooses k, the left side becomes $k(b - 1) + k = kb$ and the right side also becomes kb, so the game ends. This occurs in at most $(b - 2) + 1 = b - 1$ turns, as desired.

2/2/21. Alice has three daughters, each of whom has two daughters; each of Alice's six granddaughters has one daughter. How many sets of women from the family of 16 can be chosen such that no woman and her daughter are both in the set? (Include the empty set as a possible set.)

We proceed by cases. If Alice is in the set, then we cannot choose any of her daughters, and for each of the 6 granddaughters, we can choose either that granddaughter, that granddaughter's daughter, or neither, for a total of 3 choices per granddaughter. Thus, there are $3^6 = 729$ possible sets that include Alice.

If Alice is not in the set, then we view Alice's three daughters as matriarchs of three disjoint identical family trees. For each of her three daughters, we can choose (a) $2^2 = 4$ subsets that include the daughter and any number of that daughter's grandchildren, or (b) $3^2 = 9$ subsets that do not include the daughter, but include her children and/or grandchildren (for each daughter, we can take her, her daughter, or neither). This gives $4 + 9 = 13$ subsets associated to each of Alice's three daughters, and thus there are $(13)^3 = 2197$ subsets that do not include Alice.

This gives a total of $729 + 2197 = \boxed{2926}$ allowed subsets.

3/2/21. Prove that if a and b are positive integers such that $a^2 + b^2$ is a multiple of 7^{2009}, then ab is a multiple of 7^{2010}.

We first construct the following chart of squares modulo 7:

k	0	1	2	3	4	5	6
k^2	0	1	4	2	2	4	1

The only pair of squares modulo 7 that can add to 0 (mod 7) is $0 + 0$. Thus, $a^2 \equiv b^2 \equiv 0 \pmod 7$. This means that $a = 7a_1$ and $b = 7b_1$ for some positive integers a_1 and b_1. But then $(7a_1)^2 + (7a_2)^2$ is a multiple of 7^{2009}, so $a_1^2 + a_2^2$ is a multiple of 7^{2007}. By the same reasoning, $a_1 = 7a_2$ and $b_1 = 7b_2$ for some positive integers a_2 and b_2; hence $a = 7^2 a_2$ and $b = 7^2 b_2$. But then $a_2^2 + b_2^2$ is a multiple of 7^{2005}, so we can repeat the process.

Inductively, we get that $a = 7^{1005} a_{1005}$ and $b = 7^{1005} b_{1005}$ for some positive integers a_{1005} and b_{1005}, so that $ab = 7^{2010}(a_{1005} b_{1005})$, as desired.

4/2/21. The Rational Unit Jumping Frog starts at $(0,0)$ on the Cartesian plane, and each minute jumps a distance of exactly 1 unit to a point with rational coordinates.

(a) Show that it is possible for the frog to reach the point $\left(\frac{1}{5},\frac{1}{17}\right)$ in a finite amount of time.

(b) Show that the frog can never reach the point $\left(0,\frac{1}{4}\right)$.

(a) We use the Pythagorean triples $\{3,4,5\}$ and $\{8,15,17\}$ to construct the path from $(0,0)$ to $\left(\frac{1}{5},\frac{1}{17}\right)$ shown below:

$$(0,0) \to \left(\frac{3}{5},\frac{4}{5}\right) \to \left(\frac{6}{5},0\right) \to \left(\frac{1}{5},0\right) \to \left(\frac{1}{5}-\frac{15}{17},-\frac{8}{17}\right) \to \left(\frac{1}{5},-\frac{16}{17}\right) \to \left(\frac{1}{5},\frac{1}{17}\right).$$

(b) Suppose the frog jumps such that its position changes by (r,s), where r and s are rational. Let c be the least common denominator of r and s, so that $r=\frac{a}{c}$, $s=\frac{b}{c}$ for some integers a and b with $\gcd(a,b)=1$. The length of each jump is 1, so we have $\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2=1$, which gives $a^2+b^2=c^2$.

If a and b are both odd, then $c^2=a^2+b^2\equiv 2 \pmod 4$. Since no square is congruent to 2 mod 4, we conclude that a and b cannot both be odd. We cannot have a and b both even because $\gcd(a,b)=1$. Therefore, one of a and b is even and the other is odd, so $c^2=a^2+b^2$ is odd, which means c is odd.

Letting the i^{th} jump change the frog's location by $\left(\frac{a_i}{c_i},\frac{b_i}{c_i}\right)$, the frog's location after k jumps is

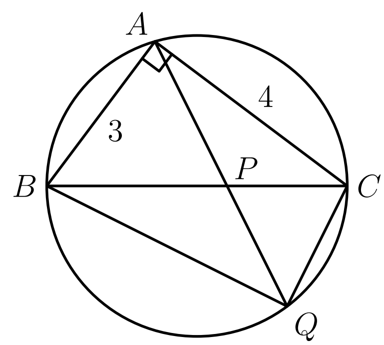
$$\left(\sum_{i=1}^{k}\frac{a_i}{c_i},\sum_{i=1}^{k}\frac{b_i}{c_i}\right).$$

Since all of the c_i are odd, the denominator of each of these sums is odd. Therefore, the frog cannot reach a point with an even denominator. Specifically, the frog cannot reach $\left(0,\frac{1}{4}\right)$.

5/2/21. Let ABC be a triangle with $AB = 3$, $AC = 4$, and $BC = 5$, let P be a point on $\overline{BC}$, and let Q be the point (other than A) where the line through A and P intersects the circumcircle of ABC. Prove that

$$PQ < \frac{25}{4\sqrt{6}}.$$

Consider the triangle ABC. Points A, C, Q, and B are concyclic, so triangles APC and BPQ are similar, which means $AC/BQ = PC/PQ$. Likewise, triangles ABP and CQP are similar, so $AB/CQ = PB/PQ$. Then $PQ \cdot AC = BQ \cdot CP$ and $PQ \cdot AB = BP \cdot CQ$. Taking the product of these equations, we get



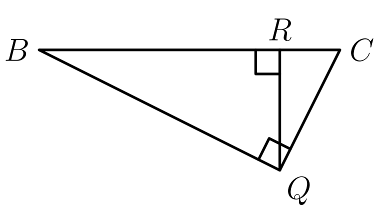
$$PQ^2 \cdot AB \cdot AC = BP \cdot CP \cdot BQ \cdot CQ$$
$$PQ^2 \cdot 3 \cdot 4 = BP \cdot CP \cdot BQ \cdot CQ$$

By the AM-GM Inequality,

$$BP \cdot CP \le \left(\frac{BP + CP}{2}\right)^2 = \frac{25}{4},$$

which tells us

$$PQ^2 \le \frac{25}{48} BQ \cdot CQ.$$



Now we must maximize $BQ \cdot CQ$. This value is the twice area of the triangle BCQ, since Q is also a right angle. Let R be the base of the altitude of BCQ from Q. Then

$$BQ \cdot CQ = 2[BCQ] = BC \cdot RQ = 5RQ.$$

Since Q lies on the unit circle and R lies on a diameter, the length of RQ is maximized when R is the center of the circle, making RQ the radius, so

$$BQ \cdot CQ = 5RQ \le 5 \cdot \frac{5}{2} = \frac{25}{2}$$

This tells us that

$$PQ^2 \le \frac{25}{48} BQ \cdot CQ \le \frac{25}{48} \cdot \frac{25}{2},$$

or

$$PQ \leq \frac{25}{4\sqrt{6}}.$$

Finally notice that we used two inequalities in this argument. We maximized $BP \cdot CP$ via the AM-GM Inequality, which is uniquely maximized at $BP = CP$, so when P is the center of the circle. We also maximized the area $[BQC]$, which is uniquely maximized when AQ is the angle bisector of A. However the angle bisector can only pass through the center of the circle if $AB = AC$. Therefore we do not achieve both maxima simultaneously and the inequality is strict:

$$PQ < \frac{25}{4\sqrt{6}}.$$

Credits: Problem statements and solutions were written by USAMTS staff.
Problem 1/2/21 was submitted by Gaku Liu.
Problem 2/2/21 was submitted by Ben Dozier.
Problem 3/2/21 is based on a problem that appeared in the March 1997 issue of Crux Mathematicorum with Mathematical Mayhem.
Problem 4/2/21 is based on a problem from the 1994 Old Mutual Mathematics Olympiad and appeared in the January 1995 issue of Mathematical Digest.
Problem 5/2/21 is a simplified version of a problem that appeared in issue 4/96 of Mathematics and Informatics Quarterly.