
1/4/20. Consider a sequence $\{a_n\}$ with $a_1 = 2$ and $a_n = \dfrac{a_{n-1}^2}{a_{n-2}}$ for all $n \geq 3$. If we know that a_2 and a_5 are positive integers and $a_5 \leq 2009$, then what are the possible values of a_5?

Since a_1 and a_2 are positive integers, all of the subsequent terms must be positive. Divide both sides of the recursion by a_{n-1} to get

$$\frac{a_n}{a_{n-1}} = \frac{a_{n-1}}{a_{n-2}}.$$

Thus, the ratio of consecutive terms is constant, and the sequence is a geometric sequence.

If $a_2 = x$, then the ratio between consecutive terms is $x/2$. Hence $a_5 = 2\left(\dfrac{x}{2}\right)^4 = \dfrac{x^4}{8}$. For this to be an integer, given that x is an integer, it is necessary and sufficient that x be a multiple of 2.

The inequality $a_5 \leq 2009$ gives us

$$\frac{x^4}{8} \leq 2009 \quad \Leftrightarrow \quad x^4 \leq 16072.$$

Note that $11^4 < 16072 < 12^4$, so we must have $x \leq 11$. But since x must be even, we must have $x \in \{2, 4, 6, 8, 10\}$. Plugging these values of x into $a_5 = x^4/8$ gives:

$$\boxed{a_5 \in \{2, 32, 162, 512, 1250\}}$$

2/4/20. There are k mathematicians at a conference. For each integer n from 0 to 10, inclusive, there is a group of 5 mathematicians such that exactly n pairs of those 5 mathematicians are friends. Find (with proof) the smallest possible value of k.

There must be 5 mathematicians that are all friends (giving 10 pairs of friends for that group), and 5 mathematicians that all are not friends (giving 0 pairs of friends for that group). If $k \le 8$, then these conditions cannot both be simultaneously satisfied: if there are 5 mathematicians that are all friends, then any group of 5 mathematicians will contain at least 2 from the group of 5 that are all friends, so we cannot find a group of 5 with no pairs of friends.

Thus we must have $k \ge 9$. We will show that $k = 9$ is achievable.

Let A, B, C, D, E be group of 5 mathematicians that are all friends, and let W, X, Y, Z be a group that are all not friends. Further, suppose:

A is friends with W, X, Y, and Z
B is friends with W, X and Y (and not friends with Z)
C is friends with W and X (and not friends with Y and Z)
D is friends with W (and not friends with X, Y, and Z)
E is not friends with any of W, X, Y, and Z

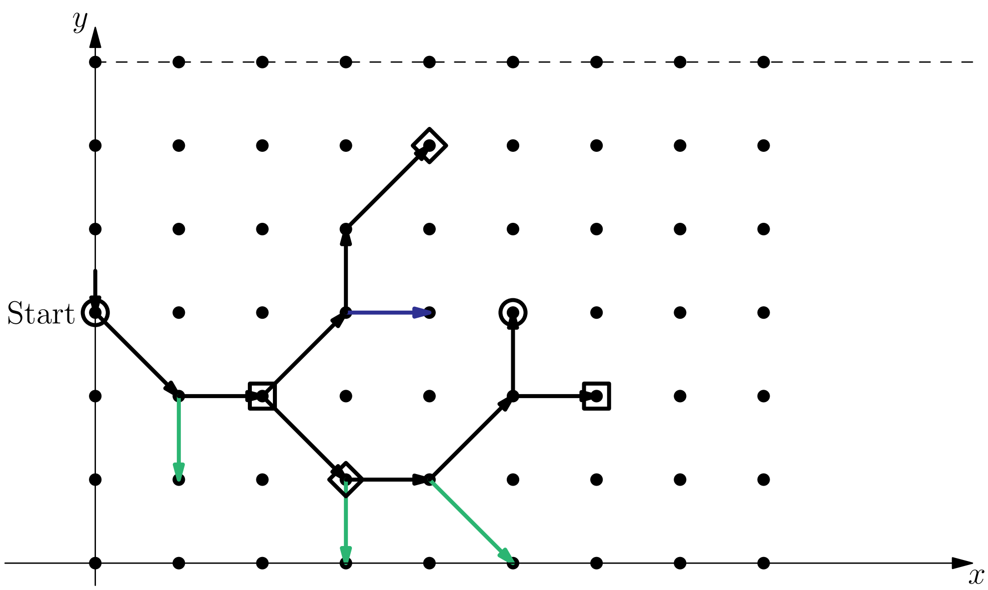
Then we have the following groups with the required exact number of friends:

Subset	Number	Pairs of friends
$\{E, W, X, Y, Z\}$	0	none
$\{D, W, X, Y, Z\}$	1	$\{D, W\}$
$\{C, W, X, Y, Z\}$	2	$\{C, W\}, \{C, X\}$
$\{B, W, X, Y, Z\}$	3	$\{B, W\}, \{B, X\}, \{B, Y\}$
$\{A, W, X, Y, Z\}$	4	$\{A, W\}, \{A, X\}, \{A, Y\}, \{A, Z\}$
$\{B, C, D, X, Z\}$	5	$\{B, C\}, \{B, D\}, \{C, D\}, \{B, X\}, \{C, X\}$
$\{B, C, D, E, Z\}$	6	all 6 pairs in $\{B, C, D, E\}$
$\{A, B, C, D, Z\}$	7	$\{A, Z\}$, all 6 pairs in $\{A, B, C, D\}$
$\{A, B, C, D, Y\}$	8	$\{A, Y\}, \{B, Y\}$, all 6 pairs in $\{A, B, C, D\}$
$\{A, B, C, D, X\}$	9	$\{A, X\}, \{B, X\}, \{C, X\}$, all 6 pairs in $\{A, B, C, D\}$
$\{A, B, C, D, E\}$	10	all 10 pairs in $\{A, B, C, D, E\}$

Thus the smallest possible value of k is $\boxed{k = 9}$.

3/4/20. A particle is currently at the point $(0, 3.5)$ on the plane and is moving towards the origin. When the particle hits a lattice point (a point with integer coordinates), it turns with equal probability 45° to the left or to the right from its current course. Find the probability that the particle reaches the x-axis before hitting the line $y = 6$.

Note that the direction of the first move is irrelevant because of the symmetry. After that, we can sketch the possibilities:



The green arrows are guaranteed wins. If the particle follows the blue arrow ending at $(4, 3)$, then the probability of winning from there is $\frac{1}{2}$, by symmetry.

Let:
p be the probability of winning from the start circle at $(0, 3)$
q be the probability of winning from the square at $(2, 2)$
r be the probability of winning from the diamond at $(3, 1)$

We then note, by symmetry, that:
the probability of winning from the circle at $(5, 3)$ is $1 - p$
the probability of winning from the square at $(6, 2)$ is q
the probability of winning from the diamond at $(4, 5)$ is $1 - r$

Therefore, we can write the following system of equations:

$$p = \frac{1}{2} + \frac{1}{2}q,$$

$$q = \frac{1}{8} + \frac{1}{2}r + \frac{1}{4}(1-r),$$

$$r = \frac{3}{4} + \frac{1}{8}q + \frac{1}{8}(1-p).$$

We can clear the denominators and collect terms:

$$\begin{aligned} 2p &= 1 + q, \\ 8q &= 3 + 2r, \\ 8r &= 7 - p + q. \end{aligned}$$

Substituting the 3rd equation into the 2nd equation gives:

$$\begin{aligned} 2p &= 1 + q, \\ 31q &= 19 - p. \end{aligned}$$

So the first equation becomes

$$62p = 31 + 31q = 50 - p,$$

hence $63p = 50$ and $p = \boxed{\dfrac{50}{63}}$.

4/4/20. Find, with proof, all functions f defined on nonnegative integers taking nonnegative integer values such that

$$f(f(m)+f(n)) = m+n$$

for all nonnegative integers m, n.

Let $a = f(0)$. Plugging in $m = n = 0$ to the equation gives

$$0 = m+n = f(f(m)+f(n)) = f(2f(0)) = f(2a).$$

So $f(2a) = 0$. Then, plugging in $m = n = 2a$ gives

$$4a = m+n = f(f(m)+f(n)) = f(f(2a)+f(2a)) = f(0+0) = f(0) = a.$$

So $4a = a$, hence $a = 0$. Thus $f(0) = 0$.

Now, plugging in $n = 0$ for an arbitrary m gives

$$m = m+0 = f(f(m)+f(0)) = f(f(m)+0) = f(f(m)),$$

so $f(f(m)) = m$ for all m. In particular, apply f to both sides of the original equation to get

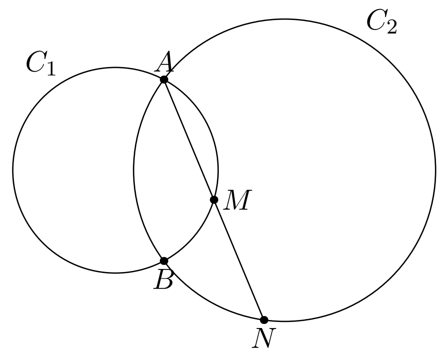
$$f(m)+f(n) = f(f(f(m)+f(n))) = f(m+n).$$

In particular, letting $n = 1$ gives $f(m+1) = f(m) + f(1)$.

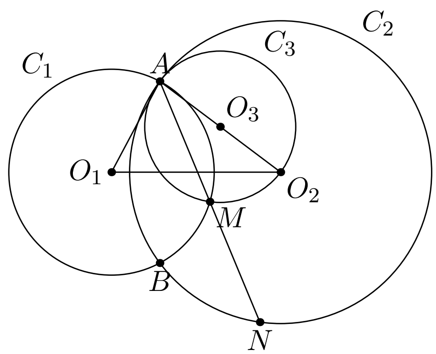
Let $f(1) = b$, so that (by a trivial induction) we have $f(m) = mb$ for all nonnegative integers m. But $m = f(f(m)) = f(mb) = mb^2$, so we must have $b^2 = 1$, hence $b = 1$.

Therefore, the only function that satisfies the functional equation is $\boxed{f(m) = m \text{ for all } m}$.

5/4/20. A circle C_1 with radius 17 intersects a circle C_2 with radius 25 at points A and B. The distance between the centers of the circles is 28. Let N be a point on circle C_2 such that the midpoint M of chord AN lies on circle C_1. Find the length of AN.



Let C_3 be the image of C_2 under a dilation through A by a factor of $1/2$. Let O_1, O_2, O_3 be the centers of C_1, C_2, C_3, respectively, so O_3 is the midpoint of AO_2.



Then M is the image of N under this dilation. However, M also lies on C_1, so M is the intersection of C_1 and C_3, other than A.

Let P be the intersection of O_1O_3 and AM. Since AM is a common chord of circles C_1 and C_3, $AM \perp O_1O_3$, so AP is the height from vertex A to base O_1O_3 in triangle AO_1O_3.

Let $\theta = \angle O_1AO_3$. Note that $AO_1 = 17$, $AO_2 = 25$, and $O_1O_2 = 28$, so by the Law of Cosines,

$$\cos\theta = \frac{17^2 + 25^2 - 28^2}{2 \cdot 17 \cdot 25} = \frac{13}{85}.$$

Then

$$\sin^2\theta = 1 - \frac{13^2}{85^2} = \frac{7056}{85^2} = \frac{84^2}{85^2},$$

so

$$\sin\theta = \frac{84}{85}.$$

(Since $0 < \theta < \pi$, we take the positive root.)

Then

$$[O_1AO_3] = \frac{1}{2}AO_1 \cdot AO_3 \sin\theta = \frac{1}{2} \cdot 17 \cdot \frac{25}{2} \cdot \frac{84}{85} = 105,$$

and again by the Law of Cosines,

$$\begin{aligned}(O_1O_3)^2 &= (AO_1)^2 + (AO_3)^2 - 2AO_1 \cdot AO_3 \cos\theta \\ &= 17^2 + \frac{25^2}{4} - 2 \cdot 17 \cdot \frac{25}{2} \cdot \frac{13}{85} \\ &= 289 + \frac{625}{4} - 65 \\ &= \frac{1521}{4} \\ &= \frac{39^2}{2^2},\end{aligned}$$

hence $O_1O_3 = \dfrac{39}{2}$.

Therefore,

$$AP = \frac{2[O_1AO_3]}{O_1O_3} = \frac{2 \cdot 105}{39/2} = \frac{140}{13}.$$

Finally, P is the midpoint of AM, and M is the midpoint of AN, so

$$AN = 4AP = \boxed{\frac{560}{13}}.$$

Credits: All problems and solutions are by USAMTS staff.