USA Mathematical Talent Search

Round 3 Solutions

Year 20 — Academic Year 2008–2009

www.usamts.org

1/3/20. Let S be the set of all 10-digit numbers (which by definition may not begin with 0) in which each digit 0 through 9 appears exactly once. For example, the number 3,820,956,714 is in S. A number n is picked from S at random. What is the probability that n is divisible by 11?

Let the number n be represented as $abcdefghij$, where each letter represents a digit. Then n is divisible by 11 if and only if the "odd-digit" sum $a + c + e + g + i$ and the "even-digit" sum $b + d + f + h + j$ differ by a multiple of 11. However, since we know that

$$a+b+c+d+e+f+g+h+i+j = 0+1+2+3+4+5+6+7+8+9 = 45,$$

the only possibility is that one sum (the odds or the evens) is 17 and the other is 28. We can now list the sets of 5 distinct digits that sum to 28 (the other 5 digits will necessarily have to sum to 17):

$$\begin{array}{llll} 9+8+7+4+0 & 9+8+7+3+1 & 9+8+6+5+0 & 9+8+6+4+1 \\ 9+8+6+3+2 & 9+8+5+4+2 & 9+7+6+5+1 & 9+7+6+4+2 \\ 9+7+5+4+3 & 8+7+6+5+2 & 8+7+6+4+3 & \end{array}$$

Thus, there are 11 ways to split the digits into one group of five that sum to 28 and one group of five that sum to 17. For each of these 11 ways, we have 2 choices which digits get put into the odd positions and which get put into the even positions, and then 5! ways to arrange the five digits in each group. However, exactly 1/10 of these will end up with 0 as the leading digit, which is not allowed. Therefore, there are

$$11 \cdot 2 \cdot (5!)^2 \cdot \frac{9}{10} = \frac{99}{5} \cdot (5!)^2$$

numbers in S that are divisible by 11. There are $9 \cdot 9!$ numbers in S, so the probability is

$$\frac{99 \cdot (5!)^2}{5 \cdot 9 \cdot 9!} = \frac{11 \cdot 4!}{9 \cdot 8 \cdot 7 \cdot 6} = \boxed{\frac{11}{126}}.$$

USA Mathematical Talent Search

Round 3 Solutions

Year 20 — Academic Year 2008–2009

www.usamts.org

2/3/20. Two players are playing a game that starts with 2009 stones. The players take turns removing stones. A player may remove exactly 3, 4, or 7 stones on his or her turn, except that if only 1 or 2 stones are remaining then the player may remove them all. The player who removes the last stone wins. Determine, with proof, which player has a winning strategy, the first or the second player.

In a *winning position*, there exists a move that either wins the game immediately or leaves the opponent in a losing position. In a *losing position*, every move results in a winning position (for the other player). We can start with a small number of stones and work backwards to classify the winning and losing positions. We will list all positions with 20 or fewer stones:

# of stones	Type of position	Winning move (if any)
1	Winning	take 1
2	Winning	take 2
3	Winning	take 3
4	Winning	take 4
5	Losing	must leave 1 or 2
6	Losing	must leave 2 or 3
7	Winning	take 7
8	Winning	take 3
9	Winning	take 3
10	Winning	take 4
11	Losing	must leave 4, 7, or 8
12	Winning	take 7
13	Winning	take 7
14	Winning	take 3
15	Winning	take 4
16	Losing	must leave 9, 12, or 13
17	Losing	must leave 10, 13, or 14
18	Winning	take 7
19	Winning	take 3
20	Winning	take 3

Clearly we do not want to continue doing this up to 2009, but we notice that the pattern repeats every 11 stones. Specifically, we note that the position type (winning or losing) only depends on the number of stones modulo 11:

# of stones (mod 11)	Type of position	Winning move (if any)
0	Losing	must leave 4, 7, or 8
1	Winning	take 7 to leave 5 (or take 1 to win)
2	Winning	take 7 to leave 6 (or take 2 to win)
3	Winning	take 3 to leave 0
4	Winning	take 4 to leave 0
5	Losing	must leave 1, 2, or 9
6	Losing	must leave 2, 3, or 10
7	Winning	take 7 to leave 0
8	Winning	take 3 to leave 5
9	Winning	take 3 to leave 6
10	Winning	take 4 to leave 6

(Note that that stones that are "left" in the above table indicates the number of stones modulo 11.)

Again, we note that from each winning position, there exists a move to a losing position, and from each losing position, all moves go to winning positions.

Since $2009 = (182)(11) + 7 \equiv 7 \pmod{11}$, the $\boxed{\text{first player wins}}$ by taking 7 stones to leave a multiple of 11.

3/3/20. Let a, b, c be three positive integers such that

$$(\mathrm{lcm}(a,b))(\mathrm{lcm}(b,c))(\mathrm{lcm}(c,a)) = (abc)\gcd(a,b,c),$$

where "lcm" means "least common multiple" and "gcd" means "greatest common divisor." Given that no quotient of any two of a, b, c is an integer (that is, none of a, b, c is an integer multiple of any other of a, b, c), find the minimum possible value of $a + b + c$.

Let p be any prime, and suppose that p^r, p^s, p^t are the maximum prime powers that are factors of a, b, c, respectively. Without loss of generality, suppose that $r \le s \le t$. Then, isolating the powers of p in our given equation gives us

$$(p^s)(p^t)(p^t) = p^{r+s+t}p^r,$$

so $s + 2t = 2r + s + t$, giving $t = 2r$.

Therefore, for any prime that appears as a factor in a, b, c, the greatest exponent of that prime (in a, b, or c) must be exactly twice the smallest exponent of that same prime (in a, b, or c). In particular, if any prime divides one of a, b, and c, then it must divide all of them.

We cannot have a, b, c all be a power of the same prime, since then clearly the smallest one would divide the largest one.

Suppose that each of a, b, c are divisible by exactly two primes p and q. We cannot have one of a, b, c have the largest power of both primes simultaneously, since otherwise the other two numbers would divide it. So without loss of generality, suppose that a has the largest power of p and b has the largest power of q. Since c does not divide a or b, we must have that c has a higher power of q than a and a higher power of p than b. So we cannot have $a = p^2q$ and $b = pq^2$, since this would imply $c = p^2q^2$ and now a and b each divide c. The next smallest possibility is $a = p^4q^2$, $b = p^2q^4$, and $c = p^3q^3$. Plugging in the smallest possible primes $p = 2$ and $q = 3$ gives

$$a = 2^4 3^2 = 144,\ b = 2^2 3^4 = 324,\ c = 2^3 3^3 = 216,$$

so the sum is $a + b + c = 684$. This is the smallest sum with two primes.

Now suppose that each of a, b, c are divisible by exactly three primes p,q,r. The smallest possibility is if, for each prime, two of the numbers are divisible by the prime and the third is divisible by the prime squared; that is, $a = p^2qr$, $b = pq^2r$, and $c = pqr^2$. Plugging in the smallest primes $p = 2$, $q = 3$, and $r = 5$ gives

$$a = 2^2 \cdot 3 \cdot 5 = 60,\ b = 2 \cdot 3^2 \cdot 5 = 90,\ c = 2 \cdot 3 \cdot 5^2 = 150,$$

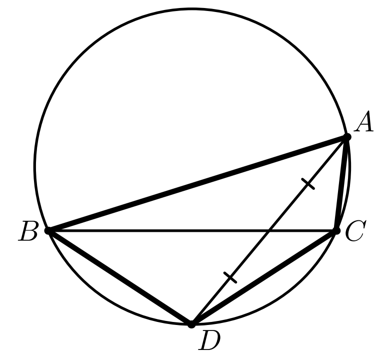
giving a sum of $a+b+c = 60+90+150 = 300$. This is the smallest sum with three primes, and is the smallest sum we've found so far.

If there are 4 or more primes, then each number must be at least $2 \cdot 3 \cdot 5 \cdot 7 = 210$, so the sum must be at least 630, which is larger than our previous example.

So the smallest possible sum is $\boxed{300}$.

4/3/20. Given a segment $\overline{BC}$ in plane $\mathcal{P}$, find the locus of all points A in $\mathcal{P}$ with the following property:

There exists *exactly one* point D in $\mathcal{P}$ such that $ABDC$ is a cyclic quadrilateral and $\overline{BC}$ bisects $\overline{AD}$, as shown at right.



We set up coordinates: let $B = (-1, 0)$ and $C = (1, 0)$. Let $A = (x, y)$ be a point in our locus. Let $E = (s, 0)$ be the point of intersection of $\overline{AD}$ and $\overline{BC}$, with $-1 < s < 1$. Then, since E is the midpoint of $\overline{AD}$, we get $D = (2s - x, -y)$. Since $ABDC$ is cyclic, we use Power of a Point at the point E to get the equation $(EB)(EC) = (EA)(ED)$; after plugging in the coordinates, this gives

$$(1 - s)(1 + s) = (s - x)^2 + y^2.$$

Simplifying gives

$$1 - s^2 = s^2 - 2xs + x^2 + y^2,$$

which gives the quadratic in s:

$$2s^2 - 2xs + x^2 + y^2 - 1 = 0.$$

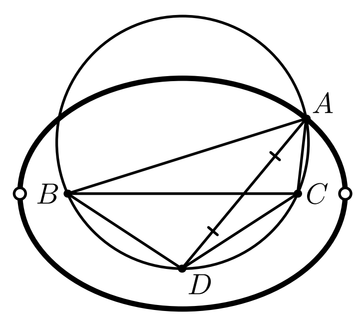
Because the point D is uniquely determined, this equation must have exactly one solution in s, so its discriminant (as a quadratic in s) must be 0. That is,

$$4x^2 - 8(x^2 + y^2 - 1) = 0.$$

Simplifying this expression gives

$$\frac{x^2}{2} + y^2 = 1.$$

So the locus of A is the set of solutions (x, y) to the above equation such that ABC is a nondegenerate triangle. Thus, we get an ellipse, centered at $(0, 0)$ (the midpoint of $\overline{BC}$), with major axis along $\overline{BC}$ with length $\sqrt{2}(BC)$, and minor axis with length BC; however, the two points on the ellipse at the ends of the major axis are not part of the locus (since at these points ABC is degenerate). Note that B and C are the foci of the ellipse.



5/3/20. Let b be an integer such that $b \geq 2$, and let $a > 0$ be a real number such that $\dfrac{1}{a}+\dfrac{1}{b} > 1$. Prove that the sequence

$$\lfloor a \rfloor, \lfloor 2a \rfloor, \lfloor 3a \rfloor, \ldots$$

contains infinitely many integral powers of b. (Note that $\lfloor x \rfloor$ is defined to be the greatest integer less than or equal to x.)

We proceed by contradiction. Suppose that, for some positive integer M and all integers $m \geq M$, the number b^m is not equal to $\lfloor ka \rfloor$ for any positive integer k. This means that for any given $m \geq M$, there exists a number n such that

$$na < b^m < b^m + 1 \leq (n+1)a. \tag{1}$$

We first prove that (1) implies

$$bna < b^{m+1} < b^{m+1} + 1 \leq (bn+1)a. \tag{2}$$

Since $na < b^m$, we get $bna < b^{m+1}$ after multiplication by b, which gives the left inequality of (2).

We now prove the right half of (2). Since $b^m + 1 \leq (n+1)a$ from (1), we multiply by b to get

$$b^{m+1} + b \leq (bn+b)a. \tag{3}$$

Since $\dfrac{1}{a} + \dfrac{1}{b} > 1$, we get that $a(b-1) < b$, and by adding b^{m+1} we get

$$b^{m+1} + a(b-1) < b^{m+1} + b. \tag{4}$$

Combining (4) with (3), we get that

$$b^{m+1} < (bn+b)a - a(b-1) = (bn+1)a. \tag{5}$$

However, we cannot have $\lfloor (bn+1)a \rfloor = b^{m+1}$, since this would violate our original assumption that b^{m+1} is not of the form $\lfloor ka \rfloor$ for any positive k. Therefore, in light of (5), we must have

$$b^{m+1} + 1 \leq (bn+1)a,$$

and the right half of (2) is proved.

We now apply (2) repeatedly to get the inequalities

$$b^r na < b^{m+r} < b^{m+r} + 1 \leq (b^r n + 1)a,$$

for all integers $r \geq 1$. But, dividing by b^r, this means that

$$na < b^m \leq \left(n + \frac{1}{b^r}\right) a \tag{6}$$

for all integers $r \geq 1$. Subtracting na from (6) gives

$$0 < b^m - na \leq \frac{a}{b^r},$$

which cannot be true for all positive integers r, since $\frac{a}{b^r}$ becomes arbitrarily small as r gets sufficiently large. This is our contradiction.

Therefore, our original assumption was false, and there are infinitely many integral powers of b in the sequence $\{\lfloor na \rfloor\}$.

Credits: Problem 3/3/20 and its solution are due to Palmer Mebane, and Dr. Erin Schram suggested the version of Problem 2/3/20. All other problems and solutions are by USAMTS staff.