USA Mathematical Talent Search

Round 2 Solutions

Year 20 — Academic Year 2008–2009

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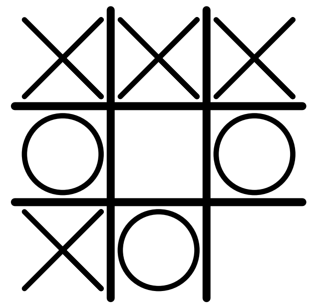
1/2/20. Sarah and Joe play a standard 3-by-3 game of tic-tac-toe. Sarah goes first and plays X, and Joe goes second and plays O. They alternate turns placing their letter in an empty space, and the first to get 3 of their letters in a straight line (across, down, or diagonal) wins. How many possible final positions are there, given that Sarah wins on her 4th move? (Don't assume that the players play with any sort of strategy; one example of a possible final position is shown at right.)



There are two cases.

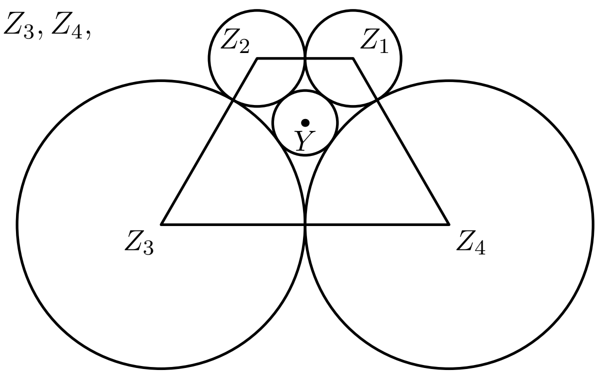
Case 1: Sarah wins in a horizontal or vertical row. There are 6 such winning rows. The three O's can be placed in any of the remaining 6 squares, except that they cannot be placed all in either of the two lines that are parallel to the winning line of X's. So there are $\binom{6}{3} - 2 = 18$ ways to place the O's. This leaves 3 squares in which to place the fourth X. Thus there are $6 \cdot 18 \cdot 3 = 324$ positions in this case.

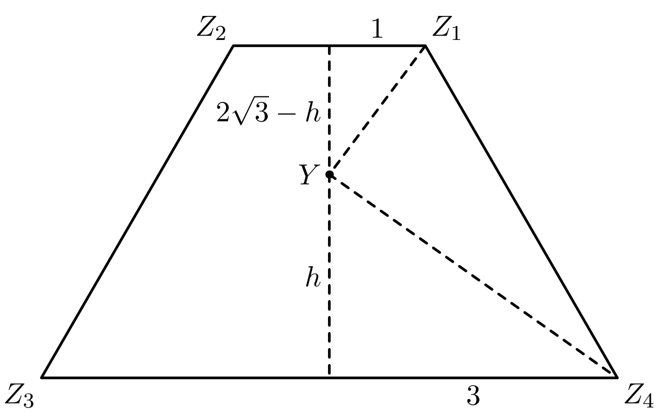
Case 2: Sarah wins in a diagonal. There are 2 such winning lines. The three O's can be placed in any of the remaining 6 squares, so there are $\binom{6}{3} = 20$ ways to place the O's. This leaves 3 squares in which to place the fourth X. Thus there are $2 \cdot 20 \cdot 3 = 120$ positions in this case.

We have a total of $324 + 120 = \boxed{444}$ possible final positions.

2/2/20. Let C_1, C_2, C_3, and C_4 be four circles, with radii 1, 1, 3, and 3, respectively, such that circles C_1 and C_2, C_2 and C_3, C_3 and C_4, and C_4 and C_1 are externally tangent. A fifth circle C is smaller than the four other circles and is externally tangent to each of them. Find the radius of C.

Denote the centers of C_1, C_2, C_3, C_4 as Z_1, Z_2, Z_3, Z_4, respectively, and the center of C as Y. Then, because $\triangle YZ_2Z_3 \cong \triangle YZ_1Z_4$ and $\triangle YZ_1Z_2$ is isosceles, we have $\angle Z_3Z_2Z_1 \cong \angle Z_4Z_1Z_2$. Thus, $Z_1Z_2Z_3Z_4$ is an isosceles trapezoid with bases $Z_1Z_2 = 2$ and $Z_3Z_4 = 6$ and legs $Z_2Z_3 = Z_4Z_1 = 4$, as shown at right. Dropping a perpendicular from Z_1 to $\overline{Z_3Z_4}$ gives a right triangle with hypotenuse 4 and base 2, so the height of the trapezoid is $2\sqrt{3}$.





The point Y lies on the segment connecting the two midpoints of the bases (because Y lies on the perpendicular bisector of each of the parallel segments $\overline{Z_1Z_2}$ and $\overline{Z_3Z_4}$). Suppose that the distance from Y to $\overline{Z_3Z_4}$ is h. Then the distance from Y to $\overline{Z_1Z_2}$ is $2\sqrt{3} - h$. Applying the Pythagorean Theorem, we get

$$YZ_1 = \sqrt{1 + (2\sqrt{3} - h)^2},$$
$$YZ_4 = \sqrt{9 + h^2}.$$

But we also know that

$$YZ_1 - 1 = YZ_4 - 3 = \text{radius of } C.$$

Thus, $YZ_1 + 2 = YZ_4$ and hence

$$\sqrt{1 + (2\sqrt{3} - h)^2} + 2 = \sqrt{9 + h^2}.$$

Square both sides to get

$$1 + h^2 - 4\sqrt{3}h + 12 + 4\sqrt{1 + (2\sqrt{3} - h)^2} + 4 = 9 + h^2.$$

Simplifying gives
$$\sqrt{1+(2\sqrt{3}-h)^2} = \sqrt{3}h - 2.$$
Squaring again gives
$$1 + h^2 - 4\sqrt{3}h + 12 = 3h^2 - 4\sqrt{3}h + 4.$$
This simplifies to $2h^2 = 9$, so $h = 3/\sqrt{2}$.

Finally, this gives
$$YZ_4 = \sqrt{9+h^2} = \sqrt{9+(9/2)} = \sqrt{(27/2)} = 3\sqrt{3/2},$$
and we conclude that
$$\text{radius of } C = YZ_4 - 3 = \boxed{3\left(\sqrt{\frac{3}{2}} - 1\right) = 3\left(\frac{\sqrt{6}}{2} - 1\right)}.$$

3/2/20. Find, with proof, all polynomials $p(x)$ with the following property:

There exists a sequence $a_0, a_1, a_2, \ldots$ of positive integers such that $p(a_0) = 1$ and for all positive integers n:

(a) $p(a_n)$ is a positive integer, and

(b) $\displaystyle\sum_{j=0}^{n-1} \frac{1}{a_j} + \frac{1}{p(a_n)} = 1.$

For any positive integer k, comparing equation (b) for $n = k$ and $n = k + 1$ gives

$$\frac{1}{p(a_k)} = \frac{1}{a_k} + \frac{1}{p(a_{k+1})}.$$

In particular,

$$\frac{1}{p(a_k)} - \frac{1}{a_k} > 0.$$

Thus, we see that $p(a_k) < a_k$ for all k, and therefore $\dfrac{p(a_k)}{a_k} < 1$. Since the a_n grow arbitrarily large (otherwise, we would have $\displaystyle\sum_{j=0}^{n-1} \frac{1}{a_j} > 1$ for large n), we have $0 < \dfrac{p(x)}{x} < 1$ for arbitrarily large x. This means that $p(x)$ must be linear or constant: if $p(x)$ had a term of degree 2 or higher, then $p(x)/x$ would have a term of degree 1 or higher, and thus would grow arbitrarily large.

Plugging in $n = 1$ to (b) gives $\dfrac{1}{a_0} + \dfrac{1}{p(a_1)} = 1$, so since both a_0 and $p(a_1)$ are positive integers, we must have $a_0 = p(a_1) = 2$. Then, since $p(a_0) = 1$, we have $p(2) = 1$. Since $p(x)$ is linear or constant, we need only determine one additional value of p in order to write the formula for $p(x)$. Since we know $p(a_1) = 2$, it suffices to find a_1 to determine $p(x)$.

Plugging in $n = 2$ gives

$$\frac{1}{2} + \frac{1}{a_1} + \frac{1}{p(a_2)} = 1,$$

so, since a_1 and $p(a_2)$ must be positive integers, we must have one of the following cases.

Case 1: $a_1 = 3$ and $p(a_2) = 6$. Then, using $p(a_1) = 2$, we have $p(3) = 2$, and hence $p(x) = x - 1$. We can verify that this works with the sequence $a_0 = 2$ and $a_n = 1 + \displaystyle\prod_{i=0}^{n-1} a_i$.

Case 2: $a_1 = 4$ and $p(a_2) = 4$. Then, using $p(a_1) = 2$, we have $p(4) = 2$, and hence $p(x) = x/2$. We can verify that this works with the sequence $a_n = 2^{n+1}$.

Case 3: $a_1 = 6$ and $p(a_2) = 3$. Then, using $p(a_1) = 2$, we have $p(6) = 2$, and hence $p(x) = (x/4) + 1/2$. However, by $p(a_2) = 3$, we have $a_2 = 10$, and plugging $n = 3$ into (b) gives

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \frac{1}{p(a_3)} = 1,$$

giving $p(a_3) = 30/7$, which is not an integer.

So the only valid polynomials are $\boxed{x - 1 \text{ and } x/2}$.

4/2/20. Find, with proof, the largest positive integer k with the following property:

There exists a positive number N such that N is divisible by all but three of the integers $1, 2, 3, \ldots, k$, and furthermore those three integers (that don't divide N) are consecutive.

Lemma 1: The three consecutive non-divisors must each be a positive power of a prime number.

Proof: On the contrary, suppose that ab is one of the non-divisors of N, where a and b are each greater than 1 and have no prime factors in common. Then either a or b does not divide N; suppose without loss of generality that a does not divide N. We then must have $ab - 2 = a$ (since a and ab must be two of three consecutive numbers), so $a(b-1) = 2$, but this means that $a = b = 2$, contradicting the assumption that a and b had no prime factors in common. □

Lemma 2: $7, 8, 9$ are the three greatest consecutive positive integers that are each a positive power of a prime.

Proof: Other than $2, 3, 4$, there is no sequence of three consecutive such integers in which the middle one is odd, because the only powers of 2 that are 2 apart are 2 and 4. So any greater sequence of three consecutive integers must be of the form

$$2^n - 1, 2^n, 2^n + 1$$

for some positive integer $n \geq 2$. Setting $n = 2$ gives the sequence $3, 4, 5$. If $n > 2$, then we cannot have n even, since otherwise

$$2^n - 1 = (2^{(n/2)} - 1)(2^{(n/2)} + 1)$$

cannot be the power of an odd prime. Therefore we must have n odd. Also, any three consecutive integers must contain a multiple of 3, so one of our numbers must be a power of 3. And, since $2^n \equiv 2 \pmod 3$ for odd n, we must have $2^n + 1 = 3^m$ for some positive integer m. Also, since $2^n \equiv 0 \pmod 4$ for $n \geq 2$, we must have m even (so that $3^m \equiv 1 \pmod 4$). But then we can factor

$$2^n = 3^m - 1 = (3^{(m/2)} - 1)(3^{(m/2)} + 1),$$

which can only be a power of 2 if the two factors on the right side above (which differ by 2) are 2 and 4, giving $n = 3$ and $m = 2$. This gives the three consecutive integers $7, 8, 9$. □

So if N is not divisible by 7, 8, or 9, then it is possible that N might be divisible by 10, 11, 12, 13, but N cannot be divisible by 14 (since it is not divisible by 7). Thus the upper bound on k is 13. We can easily verify that $N = 4 \cdot 3 \cdot 5 \cdot 11 \cdot 13$ is divisible by all of $\{1, 2, 3, 4, 5, 6, 10, 11, 12, 13\}$ but none of $\{7, 8, 9\}$. Therefore the maximum value of k is $\boxed{13}$.

5/2/20. The set S consists of 2008 points evenly spaced on a circle of radius 1 (so that S forms the vertices of a regular 2008-sided polygon). 3 distinct points X, Y, Z in S are chosen at random. The expected value of the area of $\triangle XYZ$ can be written in the form $r\cot\left(\frac{\pi}{2008}\right)$, where r is a rational number. Find r.

Let $n = 2008$. Fix one of the points A in S. We can assume without loss of generality that A is one of the vertices of our randomly chosen triangle. Let B and C be the other two vertices so that A, B, C are in clockwise order, and let O be the center of the n-sided polygon.

Then

$$\angle AOB = \frac{2\pi i}{n}, \quad \angle BOC = \frac{2\pi j}{n}, \quad \angle COA = \frac{2\pi k}{n},$$

where i, j, k are positive integers such that

$$1 \le i \le n-2, \quad 1 \le j \le n-1-i, \quad k = n-i-j.$$

Note that the angles sum to 2π and the bounds ensure that i, j, k are all positive.

The angles of triangle ABC are $\frac{\pi i}{n}$, $\frac{\pi j}{n}$, and $\frac{\pi k}{n}$. We use the result that a triangle with angles α, β, γ and circumradius R has area

$$2R^2(\sin\alpha)(\sin\beta)(\sin\gamma).$$

(This formula is an immediate consequence of the Law of Sines: it can be proved by using the general area formula $\frac{1}{2}ab\sin\gamma$, where a and b are the lengths of the sides adjacent to angle γ, and then using $(a/\sin\alpha) = (b/\sin\beta) = 2R$.) We have $R = 1$ and $\alpha = \pi i/n$, $\beta = \pi j/n$, and $\gamma = \pi k/n$, so our area is:

$$\text{Area of } ABC = 2(\sin\alpha)(\sin\beta)(\sin\gamma).$$

A double use of the product-to-sum formulas, and the fact that $\alpha + \beta + \gamma = \pi$, gives

$$\begin{aligned}\text{Area of } ABC &= 2(\sin\alpha)(\sin\beta)(\sin\gamma)\\ &= \frac{1}{2}(\sin(-\alpha+\beta+\gamma) + \sin(\alpha-\beta+\gamma) + \sin(\alpha+\beta-\gamma) - \sin(\alpha+\beta+\gamma))\\ &= \frac{1}{2}(\sin(\pi-2\alpha) + \sin(\pi-2\beta) + \sin(\pi-2\gamma) + \sin(\pi))\\ &= \frac{1}{2}(\sin 2\alpha + \sin 2\beta + \sin 2\gamma),\end{aligned}$$

Therefore,

$$\text{Area of } ABC = \frac{1}{2}\left(\sin\frac{2\pi i}{n} + \sin\frac{2\pi j}{n} + \sin\frac{2\pi k}{n}\right).$$

Summing over all triangles with vertex A, we get

$$\text{Total area} = \frac{1}{2}\sum_{i=1}^{n-2}\sum_{j=1}^{n-1-i}\left(\sin\frac{2\pi i}{n} + \sin\frac{2\pi j}{n} + \sin\frac{2\pi(n-i-j)}{n}\right).$$

We note that for any positive integer $1 \le a \le n-2$, the term $\sin\frac{2\pi a}{n}$ will appear exactly $n-1-a$ times in each position in the above sum. Therefore, we have

$$\text{Total area} = \frac{3}{2}\sum_{a=1}^{n-2}(n-1-a)\sin\frac{2\pi a}{n}.$$

Let S be the above summation (so that the total area is $\frac{3}{2}S$). Define $\omega = e^{\frac{2\pi i}{n}}$. Then S is the imaginary part of

$$z = (n-2)\omega + (n-3)\omega^2 + (n-4)\omega^3 + \cdots + \omega^{n-2}.$$

Then

$$\omega z = (n-2)\omega^2 + (n-3)\omega^3 + (n-4)\omega^4 + \cdots + \omega^{n-1}.$$

and thus

$$\begin{aligned}(1-\omega)z &= (n-2)\omega - \omega^2 - \omega^3 - \cdots - \omega^{n-2} - \omega^{n-1}\\ &= (n-1)\omega - (\omega + \omega^2 + \cdots + \omega^{n-1})\\ &= (n-1)\omega + 1.\end{aligned}$$

Thus

$$z = \frac{(n-1)\omega+1}{1-\omega} = \frac{n\omega}{1-\omega} + 1.$$

Let $\zeta = e^{\pi i/n}$. Then

$$\begin{aligned}z &= \frac{n\omega}{1-\omega} - 1\\ &= \frac{n\zeta^2}{1-\zeta^2} - 1\\ &= \frac{n\zeta}{(1/\zeta)-\zeta} - 1\\ &= \frac{n(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n})}{-2i\sin\frac{\pi}{n}} - 1\\ &= \frac{n\cos\frac{\pi}{n}}{2\sin\frac{\pi}{n}}i - \frac{n}{2} - 1.\end{aligned}$$

Thus
$$S = \operatorname{Im}(z) = \frac{n}{2}\cot\frac{\pi}{n},$$
and the total area is $\frac{3}{2}S = \frac{3}{4}n\cot\frac{\pi}{n}$.

Therefore, since there are $\binom{n-1}{2}$ choices for the points B and C, the expected value of the area is
$$\frac{\frac{3n}{4}\cot\frac{\pi}{n}}{\binom{n-1}{2}} = \frac{3n}{2(n-1)(n-2)}\cot\frac{\pi}{n}.$$

Plugging in $n = 2008$, we see that the answer is $\boxed{\dfrac{3(2008)}{2(2007)(2006)} = \dfrac{502}{(669)(1003)} = \dfrac{502}{671007}}$.

Note: Unbeknownst to the USAMTS staff and our problem reviewers at the time that the problems were being prepared, the solution to Problem 5/2/20 appeared in:

Shova KC and Anna Madra, "Randomly generated triangles whose vertices are vertices of a regular polygon", *Rose-Hulman Undergraduate Mathematics Journal*, **7**, No. 2, 2006.

A copy of this paper (which also has the solution to the continuous version of this problem) is available at `http://www.rose-hulman.edu/mathjournal/v7n2.php`

Credits: All problems and solutions were written by USAMTS staff; see special credit for Problem 5/2/20 above. An earlier version of Problem 4/2/20 was suggested by Chris Jeuell.