USA Mathematical Talent Search

Round 1 Solutions

Year 20 — Academic Year 2008–2009

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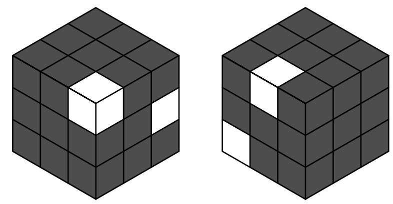
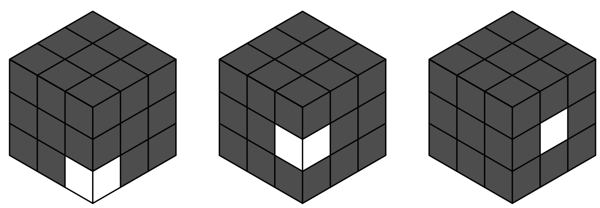
1/1/20. 27 unit cubes—25 of which are colored black and 2 of which are colored white—are assembled to form a $3 \times 3 \times 3$ cube. How many distinguishable cubes can be formed? (Two cubes are indistinguishable if one of them can be rotated to appear identical to the other. An example of two indistinguishable cubes is shown at right.)

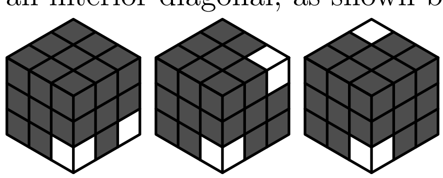


If one of the two white cubes is the central cube (that is, the cube that is not visible from the outside), then there are only 3 distinguishable places for the other white cube to be: a *corner*, an *edge*, or a *face-center*, as shown left-to-right below:

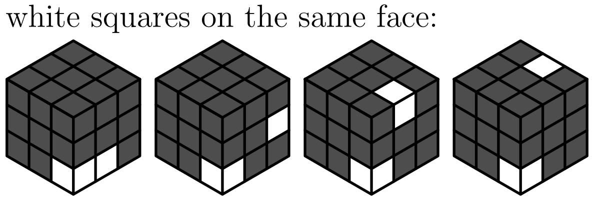


Otherwise, we have the following possibilities:

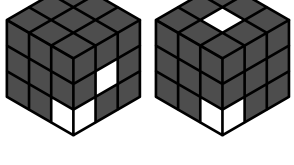
- Two corners: 3 ways—the corners lie on opposite ends of an edge, a face diagonal, or an interior diagonal, as shown below:



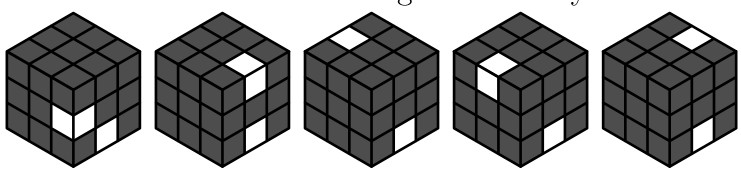
- One corner, one edge: 4 ways—the white cubes are adjacent, or give 2 non-adjacent white squares on the same face (in two different distinguishable ways), or do not give white squares on the same face:



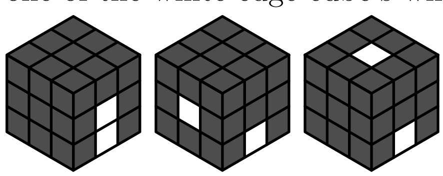
- One corner, one face-center: 2 ways—either the face-center shares a face with the corner white cube or it does not:



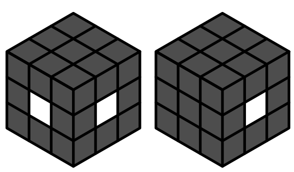
- Two edges: 5 ways—there can be white squares on the same face in two ways (either diagonally adjacent or opposite, as in the first two pictures below), or they can be on different faces in three distinguishable ways:



- One edge, one face-center: 3 ways–the face-center may be on the same face as one of the white edge cube's white squares, or the face-center is on a face adjacent to both of the white edge cube's white squares, or the face-center is on a face adjacent to only one of the white edge cube's white squares (shown left-to-right below):



- Two face-centers: 2 ways—the face-centers are either on adjacent or opposite faces:



(The last diagram has another white face-center on the face directly opposite the face with the visible white face-center.)

This gives a total of $3 + 3 + 4 + 2 + 5 + 3 + 2 = \boxed{22}$ distinguishable cubes.

2/1/20. Find all positive integers n for which it is possible to find three positive factors x, y, and z of $n-1$, with $x > y > z$, such that $x + y + z = n$.

Let $a = (n-1)/x$, $b = (n-1)/y$, and $c = (n-1)/z$, so

$$n = x + y + z = \frac{n-1}{a} + \frac{n-1}{b} + \frac{n-1}{c},$$

where a, b, and c are all positive integers, and $a < b < c$.

We start by finding bounds on a. If $a = 1$, then $x = n - 1$, so $y + z = 1$. But y and z are positive integers, so $y + z \geq 2$, giving a contradiction. Therefore, $a \geq 2$.

On the other hand, if $a \geq 3$, then $x \leq \frac{n-1}{3}$, and similarly $z < y < x \leq \frac{n-1}{3}$. So

$$n = x + y + z < \frac{n-1}{3} + \frac{n-1}{3} + \frac{n-1}{3} = n - 1,$$

giving a contradiction.

Therefore, $a = 2$. This gives $x = (n-1)/2$ and thus

$$y + z = n - x = n - \frac{n-1}{2} = \frac{n+1}{2}.$$

But recall $y = (n-1)/b$ and $z = (n-1)/c$, so we have

$$\frac{n+1}{2} = \frac{n-1}{b} + \frac{n-1}{c}. \tag{$*$}$$

Now we look for bounds on b. We already know $b > a = 2$. If $b \geq 4$, then $c > b \geq 4$, and by $(*)$ we have

$$\frac{n+1}{2} = \frac{n-1}{b} + \frac{n-1}{c} < \frac{n-1}{4} + \frac{n-1}{4} = \frac{n-1}{2},$$

giving a contradiction. So $b < 4$, and combining this with $b > 2$, we must have $b = 3$.

This means that $y = (n-1)/3$, and by $(*)$ we get

$$\frac{n-1}{c} = \frac{n+1}{2} - \frac{n-1}{3} = \frac{n+5}{6}.$$

If $c \geq 6$, then

$$\frac{n+5}{6} = \frac{n-1}{c} \leq \frac{n-1}{6},$$

giving a contradiction. Therefore, $c = 4$ or $c = 5$.

If $c = 4$, then

$$\frac{n+5}{6} = \frac{n-1}{4},$$

and we get $n = 13$. We see that 13 is breakable, because $13 = 6 + 4 + 3$.

If $c = 5$, then

$$\frac{n+5}{6} = \frac{n-1}{5},$$

and we get $n = 31$. We see that 31 is breakable, because $31 = 15 + 10 + 6$.

So the possible values of n are $\boxed{\text{13 and 31}}$.

3/1/20. Let a, b, c be real numbers. Suppose that for all real numbers x such that $|x| \leq 1$, we have $|ax^2 + bx + c| \leq 100$. Determine the maximum possible value of $|a| + |b| + |c|$.

Without loss of generality, we may assume that $a \geq 0$. (If $a < 0$, then multiplying the quadratic by -1 doesn't affect the bound nor change the value of $|a| + |b| + |c|$.) Also, replacing x by $-x$ gives the function $ax^2 - bx + c$ and does not change the bound nor the value of $|a| + |b| + |c|$. So we may also, without loss of generality, assume that $b \geq 0$.

Let $f(x) = ax^2 + bx + c$. Since $f(0) = c$ and $f(1) = a + b + c$, we must have $|c| \leq 100$ and $a + b + c \leq 100$. If $c \geq 0$, then $|a| + |b| + |c| = a + b + c \leq 100$. However, if $c < 0$, then

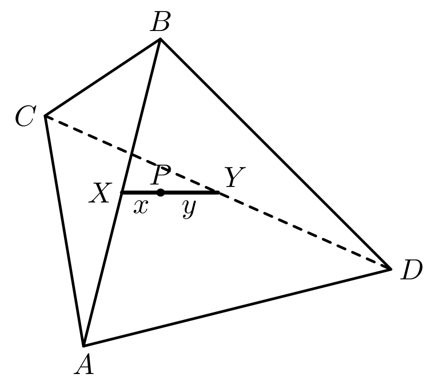
$$|a| + |b| + |c| = a + b - c = (a + b + c) - 2c \leq 100 + 200 = 300,$$

so 300 is an upper bound for the value of $|a| + |b| + |c|$.

This bound can be achieved by the function $f(x) = 200x^2 - 100$; note that the graph of this is a parabola passing through the points $(-1, 100)$, $(0, -100)$, and $(1, 100)$.

The answer is $\boxed{300}$.

4/1/20. A point P inside a regular tetrahedron $ABCD$ is such that $PA = PB = \sqrt{11}$ and $PC = PD = \sqrt{17}$. What is the side length of $ABCD$?



Solution 1: Let X be the midpoint of side $\overline{AB}$ and let Y be the midpoint of side $\overline{CD}$. Note that the locus of points P inside of $ABCD$ such that $PA = PB$ and $PC = PD$ is the line segment $\overline{XY}$ (minus its endpoints).

Let $2s$ be the side length of $ABCD$, so that $AX = CY = s$. Note that $\overline{XY}$ is a leg of the right triangle AXY with hypotenuse $AY = s\sqrt{3}$ and other leg $AX = s$, so we have $XY = s\sqrt{2}$. If we let $x = PX$ and $y = PY$, then we have the following system of equations (using the Pythagorean Theorem on right triangles PXA and PYC):

$$\begin{aligned} x^2 + s^2 &= 11, \\ y^2 + s^2 &= 17, \\ x + y &= s\sqrt{2}. \end{aligned}$$

Astute solvers might see "by inspection" that the solution to this system is $s = 3$, $x = \sqrt{2}$, $y = 2\sqrt{2}$, but we can also solve it algebraically. Using $y = s\sqrt{2} - x$ from the third equation, substitute into the second equation:

$$\begin{aligned} 17 &= (s\sqrt{2} - x)^2 + s^2 \\ &= 2s^2 - 2\sqrt{2}sx + x^2 + s^2 \\ &= 2s^2 - 2\sqrt{2}sx + 11, \end{aligned}$$

where the last step above uses $x^2 + s^2 = 11$ from our earlier system of equations. Solving for x in terms of s, we get

$$x = \frac{s^2 - 3}{s\sqrt{2}}.$$

Now we plug this into the first equation:

$$\left(\frac{s^2 - 3}{s\sqrt{2}}\right)^2 + s^2 = 11,$$

and multiply through by $2s^2$ (to clear the denominator) and expand and simplify to get

$$3s^4 - 28s^2 + 9 = 0.$$

This factors as $(3s^2 - 1)(s^2 - 9) = 0$, so $s^2 = 9$ or $s^2 = \frac{1}{3}$. Only $s^2 = 9$ works with the original system of equations, so $s = 3$.

Thus the side length of $ABCD$ is $2s = \boxed{6}$.

Solution 2: Place the tetrahedron in Cartesian 3-space using the coordinates

$$A = (a, a, a),\ B = (a, -a, -a),\ C = (-a, a, -a),\ D = (-a, -a, a).$$

Note that the side length is $2\sqrt{2}a$.

Let $P = (x, y, z)$. Then $PA = PB$ implies that $y = -z$, and $PC = PD$ implies that $y = z$, so we must have $y = z = 0$, and hence $P = (x, 0, 0)$.

Computing $PA = \sqrt{11}$ and $PC = \sqrt{17}$ gives us the system

$$\begin{aligned}(x-a)^2 + 2a^2 &= 11,\\ (x+a)^2 + 2a^2 &= 17.\end{aligned}$$

Simplifying gives $4ax = 6$, or $x = \frac{3}{2a}$. Substituting this back in gives

$$\left(\frac{3}{2a} - a\right)^2 + 2a^2 = 11,$$

and after multiplying both sides by $4a^2$, we have $(3-2a^2)^2 + 8a^4 = 44a^2$. This expands to a quadratic in a^2:

$$12a^4 - 56a^2 + 9 = 0,$$

which factors as $(2a^2 - 9)(6a^2 - 1) = 0$, so $a^2 = \frac{9}{2}$ or $a^2 = \frac{1}{6}$. But $a^2 = \frac{1}{6}$ does not give a valid solution, so $a^2 = \frac{9}{2}$, and hence $a = \frac{3}{\sqrt{2}}$.

Therefore, the side length of $ABCD$ is $2\sqrt{2}a = 2\sqrt{2}\left(\frac{3}{\sqrt{2}}\right) = \boxed{6}$.

5/1/20. Call a positive real number *groovy* if it can be written in the form $\sqrt{n}+\sqrt{n+1}$ for some positive integer n. Show that if x is groovy, then for any positive integer r, the number x^r is groovy as well.

We prove two lemmas, which we will combine to prove the overall result.

Lemma 1: If x is groovy and r is odd, then x^r is groovy.

Proof: Let $x=\sqrt{n+1}+\sqrt{n}$ and let $y=\sqrt{n+1}-\sqrt{n}$; note that $xy=1$. Also note the expansions

$$\begin{aligned} x^r &= \sum_{k=0}^{r}\binom{r}{k}\left(\sqrt{n+1}\right)^{r-k}\left(\sqrt{n}\right)^{k},\\ y^r &= \sum_{k=0}^{r}(-1)^k\binom{r}{k}\left(\sqrt{n+1}\right)^{r-k}\left(\sqrt{n}\right)^{k}. \end{aligned}$$

Since r is odd, adding x^r and y^r will cancel all of the terms in the above expansions in which k is odd, and subtracting y^r from x^r will cancel all of the terms where k is even. Thus, we see that

$$x^r+y^r = 2a\sqrt{n+1} = \sqrt{4a^2(n+1)}, \tag{1}$$
$$x^r-y^r = 2b\sqrt{n} = \sqrt{4b^2n}, \tag{2}$$

where a and b are integers. Squaring each equation in the system gives us

$$x^{2r}+2+y^{2r} = 4a^2(n+1), \tag{3}$$
$$x^{2r}-2+y^{2r} = 4b^2n. \tag{4}$$

Subtracting (4) from (3) gives $4=4a^2(n+1)-4b^2n$, so $4a^2(n+1)=4b^2n+4$.

Then, adding (1) and (2) gives us

$$2x^r = \sqrt{4a^2(n+1)}+\sqrt{4b^2n} = \sqrt{4b^2n+4}+\sqrt{4b^2n},$$

and hence $x^r=\sqrt{b^2n+1}+\sqrt{b^2n}$. Thus x^r is groovy. □

Lemma 2: If x is groovy, then x^2 is groovy.

Proof: Let $x=\sqrt{n+1}+\sqrt{n}$. We now compute:

$$\begin{aligned} x^2 &= \left(\sqrt{n+1}+\sqrt{n}\right)^2\\ &= n+1+2\sqrt{n(n+1)}+n\\ &= (2n+1)+2\sqrt{n(n+1)}\\ &= \sqrt{4n^2+4n+1}+\sqrt{4n^2+4n}. \end{aligned}$$

So x^2 is groovy. $\square$

To complete the proof, we prove that $x^{(2^a \cdot r)}$ is groovy for any nonnegative integer a and any positive odd integer r, by induction on a. The base case of the induction (where $a = 0$) is Lemma 1, and the inductive step is Lemma 2.

Credits: All problems and solutions were written by USAMTS staff. Dr. Erin Schram suggested the wording for problem 2/1/20.