5/4/18. A sequence of positive integers $(x_1, x_2, \ldots, x_{2007})$ satisfies the following two conditions:

(1) $x_n \neq x_{n+1}$ for $1 \leq n \leq 2006$, and

(2) $A_n = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$ is an integer for $1 \leq n \leq 2007$.

Find the minimum possible value of A_{2007}.

Credit This problem was a former proposal for the Canadian Mathematical Olympiad.

Comments Finding the optimal sequence is not difficut, but a high degree of rigor and careful reasoning must be employed to show conclusively that you have the minimum value. In particular, using a greedy algorithm is not sufficient. Both conditions (1) and (2) must be used effectively. *Solutions edited by Naoki Sato.*

Solution 1 by: Gaku Liu (11/FL)

We claim that the minimum value of A_n is $\lceil \frac{n+1}{2} \rceil$. This value is achieved for the sequence

$$x_n = \begin{cases} \dfrac{n+1}{2} & \text{for odd } n, \\[2ex] \dfrac{3n}{2} & \text{for even } n. \end{cases}$$

Indeed, if $n \geq 2$ is even, then $x_{n-1} = n/2$ and $x_{n+1} = (n+2)/2$, both of which are less than $x_n = 3n/2$. Hence, no two consecutive terms are equal, so condition (1) is satisfied. For even n,

$$\begin{aligned} A_n &= \frac{(x_1 + x_3 + \cdots + x_{n-1}) + (x_2 + x_4 + \cdots + x_n)}{n} \\ &= \frac{(1 + 2 + \cdots + n/2) + (3 + 6 + \cdots + n/2)}{n} \\ &= \frac{4(1 + 2 + \cdots + n/2)}{n} \\ &= \frac{4 \cdot n/2 \cdot (n+2)/2}{2n} \\ &= \frac{n+2}{2} = \left\lceil \frac{n+1}{2} \right\rceil, \end{aligned}$$

and for odd n,

$$\begin{aligned}
A_n &= \frac{(x_1 + x_3 + \cdots + x_{n-2}) + (x_2 + x_4 + \cdots + x_{n-1}) + x_n}{n} \\
&= \frac{[1 + 2 + \cdots + (n-1)/2] + [3 + 6 + \cdots + 3(n-1)/2] + (n+1)/2}{n} \\
&= \frac{4[1 + 2 + \cdots + (n-1)/2] + (n+1)/2}{n} \\
&= \frac{4 \cdot 1/2 \cdot (n-1)/2 \cdot (n+1)/2 + (n+1)/2}{n} \\
&= \frac{n+1}{2} = \left\lceil \frac{n+1}{2} \right\rceil .
\end{aligned}$$

We now prove this is the minimum through induction. It is true for $n = 1$, because the minimum of A_1 is $1 = \lceil \frac{1+1}{2} \rceil$. For $n = 2$, if $A_2 = 1$, then $x_1 + x_2 = 2 \Rightarrow x_1 = x_2 = 1$, which contradicts (1). Hence, the minimum of A_2 is $2 = \lceil \frac{2+1}{2} \rceil$.

Now, assume that $A_{2m} \geq \lceil \frac{2m+1}{2} \rceil = m + 1$ for some positive integer m. Let $S_n = x_1 + x_2 + \cdots + x_n$. In particular, S_n must be a multiple of n. We have $S_{2m} = 2mA_{2m} \geq 2m(m+1) = 2m^2 + 2m$. Also,

$$\begin{aligned}
& 2m^2 + m < 2m^2 + 2m < 2m^2 + 3m + 1 \\
\Rightarrow \quad & m(2m+1) < 2m^2 + 2m < (m+1)(2m+1),
\end{aligned}$$

so the least multiple of $2m + 1$ greater than $2m^2 + 2m$ is $(m+1)(2m+1)$. Since $S_{2m+1} > S_{2m} \geq 2m^2 + 2m$, we have $S_{2m+1} \geq (m+1)(2m+1)$, so

$$A_{2m+1} \geq m + 1 = \left\lceil \frac{(2m+1)+1}{2} \right\rceil .$$

Note that $2m^2 + 2m = m(2m+2)$ is a multiple of $2m+2$. The next greatest multiple of $2m + 2$ is $(m+1)(2m+2)$. Suppose that $S_{2m+2} = (m+1)(2m+2) = 2m^2 + 4m + 2$. Then

$$\begin{aligned}
& 2m^2 + 3m + 1 < 2m^2 + 4m + 2 < 2m^2 + 5m + 2 \\
\Rightarrow \quad & (m+1)(2m+1) < 2m^2 + 4m + 2 < (m+2)(2m+1),
\end{aligned}$$

so the greatest multiple of $2m + 1$ less than $2m^2 + 4m + 2$ is $(m+1)(2m+1)$. Since $S_{2m+1} < S_{2m+2} = 2m^2 + 4m + 2$, we have $S_{2m+1} \leq (m+1)(2m+1)$. But we have already shown that $S_{2m+1} \geq (m+1)(2m+1)$, so $S_{2m+1} = (m+1)(2m+1)$.

Also,

$$\begin{aligned}
& 2m^2 + 2m < 2m^2 + 3m + 1 < 2m^2 + 4m \\
\Rightarrow \quad & (m+1)2m < 2m^2 + 3m + 1 < (m+2)2m,
\end{aligned}$$

so $2m^2 + 2m$ is the greatest multiple of $2m$ less than S_{2m+1}. Since $S_{2m} < S_{2m+1} = (m+1)(2m+1)$, we have $S_{2m} \le (m+1)2m$. But $S_{2m} \ge (m+1)2m$, so $S_{2m} = (m+1)2m$. Then $x_{2m+1} = S_{2m+1} - S_{2m} = (m+1)(2m+1) - (m+1)2m = m+1$, and $x_{2m+2} = S_{2m+2} - S_{2m+1} = (m+1)(2m+2) - (m+1)(2m+1) = m+1$, which contradicts (1).

Hence, $S_{2m+2} \ge (m+2)(2m+2)$, so

$$A_{2m+1} \ge m + 2 = \left\lceil \frac{(2m+2)+1}{2} \right\rceil,$$

completing the induction. Therefore, the minimum value of A_{2007} is $\left\lceil \frac{2007+1}{2} \right\rceil = 1004$.