# USA Mathematical Talent Search

Solutions to Problem 5/3/19

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**5/3/19.** For every rational number $0 < \dfrac{p}{q} < 1$, where $p$ and $q$ are relatively prime, construct a circle with center $\left(\dfrac{p}{q}, \dfrac{1}{2q^2}\right)$ and diameter $\dfrac{1}{q^2}$. Also construct circles centered at $\left(0, \dfrac{1}{2}\right)$ and $\left(1, \dfrac{1}{2}\right)$ with diameter 1.

(a) Prove that any two such circles intersect in at most 1 point.

(b) Prove that the total area of all of the circles is $\dfrac{\pi}{4}\left(1 + \dfrac{\sum_{i=1}^{\infty} \frac{1}{i^3}}{\sum_{i=1}^{\infty} \frac{1}{i^4}}\right)$.

**Comments** Part (a) may be solved by comparing the distance between the centers of two circles to the sum of the radii. Part (b) may be solved by expressing the same sum two different ways, one involving the circles in the problem, and the other involving the infinite sums $\sum_{i=1}^{\infty} \frac{1}{i^3}$ and $\sum_{i=1}^{\infty} \frac{1}{i^4}$. *Solutions edited by Naoki Sato.*

**Solution by: Dmitri Gekhtman (11/IN)**

(a) Let $r$ and $s$ be two nonnegative integers such that $0 \le r \le s$ and either $r$ and $s$ are relatively prime or $s = 1$. For all possible values of $r$ and $s$, construct a circle with center $(\frac{r}{s}, \frac{1}{2s^2})$ and diameter $\frac{1}{s^2}$. The resulting set of circles is the same as the one described in the problem. (The parameters $r = 0$, $s = 1$ and $r = 1$, $s = 1$ specify the circles centered at $(0, \frac{1}{2})$ and $(1, \frac{1}{2})$ with diameter 1, respectively.)

Choose two distinct circles from the set of constructed circles. Let one have center $(\frac{r_1}{s_1}, \frac{1}{2s_1^2})$ and diameter $\frac{1}{s_1^2}$ and the other have center $(\frac{r_2}{s_2}, \frac{1}{2s_2^2})$ and diameter $\frac{1}{s_2^2}$. Since, for each circle, the radius is equal to the $y$-coordinate of the center, both circles are tangent to the $x$-axis and lie above it. Therefore, one circle cannot lie in the interior of the other circle. The $x$-coordinate of the center of each circle is the same as the $x$-coordinate of the intersection of the circle and the $x$-axis. Since the circles are tangent to the the $x$-axis at different points, they obviously cannot be internally tangent. Therefore, either (1) the two circles are external to each other and do not intersect at any point, (2) the two circles are externally tangent to each other and intersect at exactly one point, or (3) the two circles intersect at exactly two points.

Let $d$ be the distance between the centers of the two circles, and let $R$ be the sum of the their radii. In case (1), $d > R$. In case (2), $d = R$. In case (3), $d < R$. The distance between the centers of the two circles is

$$d = \sqrt{\left(\frac{r_1}{s_1} - \frac{r_2}{s_2}\right)^2 + \left(\frac{1}{2s_1^2} - \frac{1}{2s_2^2}\right)^2}.$$

The sum of the radii of the two circles is

$$R = \frac{1}{2s_1^2} + \frac{1}{2s_2^2}.$$

So

$$\begin{aligned}
d^2 - R^2 &= \left(\frac{r_1}{s_1} - \frac{r_2}{s_2}\right)^2 + \frac{1}{4}\left[\left(\frac{1}{s_1^2} - \frac{1}{s_2^2}\right)^2 - \left(\frac{1}{s_1^2} + \frac{1}{s_2^2}\right)^2\right] \\
&= \frac{r_1^2}{s_1^2} - \frac{2r_1r_2}{s_1s_2} + \frac{r_2^2}{s_2^2} - \frac{1}{s_1^2s_2^2} \\
&= \frac{r_1^2s_2^2 - 2r_1s_2r_2s_1 + r_2^2s_1^2 - 1}{s_1^2s_2^2} \\
&= \frac{(r_1s_2 - r_2s_1)^2 - 1}{s_1^2s_2^2}.
\end{aligned}$$

Since $r_1$, $r_2$, $s_1$, and $s_2$ are integers, $r_1s_2 - r_2s_1$ is an integer. Suppose that $r_1s_2 - r_2s_1 = 0$. Then

$$\frac{r_1}{s_1} = \frac{r_2}{s_2}.$$

But $\frac{r_1}{s_1}$ and $\frac{r_2}{s_2}$ are distinct rational numbers, so $r_1s_2 - r_2s_1 \neq 0$. Since $r_1s_2 - r_2s_1$ is a nonzero integer, $(r_1s_2 - r_2s_1)^2 \geq 1$, so

$$d^2 - R^2 = \frac{(r_1s_2 - r_2s_1)^2 - 1}{s_1^2s_2^2} \geq 0,$$

which means $d^2 \geq R^2$. Since $d$ and $R$ are positive, $d \geq R$. Therefore, the circles cannot intersect at two points. Furthermore, $d$ can equal $R$ (for example, when $r_1 = 0$, $s_1 = 1$, $r_2 = 1$, and $s_2 = 1$, $d = R = 1$). So any two circles can intersect in at most 1 point. This completes the proof.

(b) Consider all pairs of nonnegative integers $a$ and $b$ such that $0 \leq a \leq b$ and $b \neq 0$. The sum over all such pairs $(a, b)$ of $\frac{1}{b^4}$ is

$$\sum_{b=1}^{\infty}\sum_{a=0}^{b}\frac{1}{b^4} = \sum_{b=1}^{\infty}\frac{b+1}{b^4}.$$

Note that each pair $(a, b)$ can be uniquely written in the form $(nr, ns)$, where $n$ is a positive integer and $(r, s)$ is a pair of integers of the type described in part (a). Thus, we may write the sum over all such pairs $(a, b)$ of $\frac{1}{b^4}$ as the sum over all triples $(r, s, n)$ of $\frac{1}{(ns)^4}$. In other words (note that all series below are absolutely convergent),

$$\sum_{b=1}^{\infty}\frac{b+1}{b^4} = \sum_{(r,s)}\sum_{n=1}^{\infty}\frac{1}{(ns)^4} = \sum_{(r,s)}\frac{1}{s^4}\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

Therefore,

$$\sum_{(r,s)} \frac{1}{s^4} = \frac{\displaystyle\sum_{b=1}^{\infty} \frac{b+1}{b^4}}{\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}} = \frac{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^4} + \sum_{i=1}^{\infty} \frac{1}{i^3}}{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^4}} = 1 + \frac{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^3}}{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^4}}.$$

Multiplying both sides of this equation by $\frac{\pi}{4}$, we get

$$\sum_{(r,s)} \frac{\pi}{4s^4} = \frac{\pi}{4}\left(1 + \frac{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^3}}{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^4}}\right).$$

The left side of the equation is the sum over all such pairs $(r, s)$ of the area of a circle of diameter $\frac{1}{s^2}$. Hence, the total area of all of the circles is

$$\frac{\pi}{4}\left(1 + \frac{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^3}}{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^4}}\right).$$

This completes the proof.