USA Mathematical Talent Search

Solutions to Problem 5/3/16

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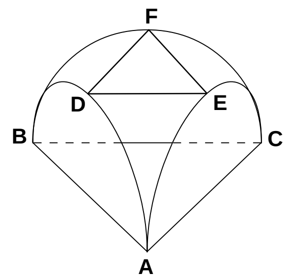
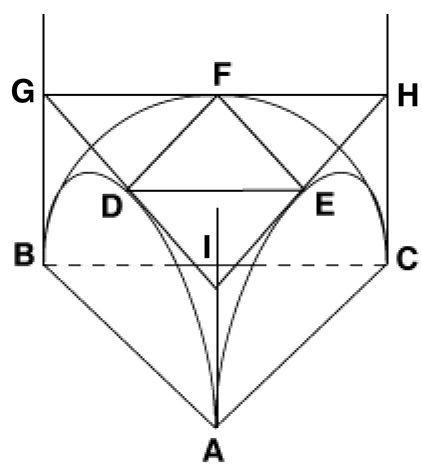
5/3/16. Consider an isosceles triangle ABC with side lengths $AB = AC = 10\sqrt{2}$ and $BC = 10\sqrt{3}$. Construct semicircles P, Q, and R with diameters AB, AC, BC respectively, such that the plane of each semicircle is perpendicular to the plane of ABC, and all semicircles are on the same side of plane ABC as shown. There exists a plane above triangle ABC that is tangent to all three semicircles P, Q, R at the points D, E, and F respectively, as shown in the diagram. Calculate, with proof, the area of triangle DEF.



Credit This problem was contributed by Professor Vladimir Fainzilberg of the Department of Chemistry at the C. W. Post Campus of Long Island University.

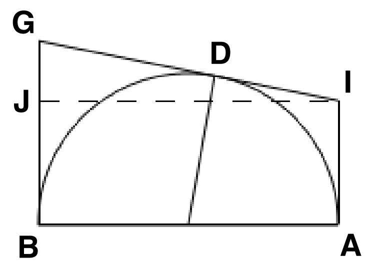
Comments The most common mistake in this problem was asserting that points D and E are at the midpoints of the their respective semi-circles. Some students successfully slogged through this problem with calculus or coordinates. Lawrence Chan shows us a geometric solution and Tony Liu mixes in a little trigonometry. *Solutions edited by Richard Rusczyk*

Solution 1 by: Lawrence Chan (11/IL)



We begin this problem by first drawing lines through A, B, and C perpendicular to the plane of triangle ABC. Then, we draw the lines where the plane containing triangle DEF (that is, the plane tangent to all three circles) intersects each of the three circles' plane (labeling intersection points as shown above). Because $\overline{BG}$ and $\overline{CH}$ were drawn perpendicular to the plane of triangle ABC, and because of symmetry due to the two closest circles being congruent, we know that $BGHC$ is a rectangle. Thus, $BG = CH = 5\sqrt{3}$.

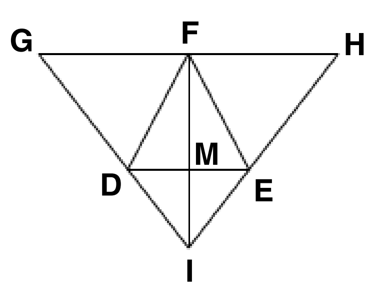
We now turn our focus to the plane of the circle with diameter $\overline{AB}$.



Since $\overline{BG}$, $\overline{GD}$, $\overline{DI}$, and $\overline{IA}$ are all tangent to the semicircle, we know that $BG = GD = 5\sqrt{3}$ and $DI = IA$. Let us call the length of $IA = x$. If we draw a line parallel to $\overline{BA}$ and passing through I, we form a right triangle GJI and a rectangle $BJIA$. Thus, $JB = IA = DI = x$ and $JI = BA$. Using the Pythagorean Theorem on GJI gives us the following result:

$$\begin{aligned}
GJ^2 + JI^2 &= GI^2 \\
(GB - JB)^2 + BA^2 &= (GD + DI)^2 \\
(5\sqrt{3} - x)^2 + (10\sqrt{2})^2 &= (5\sqrt{3} + x)^2 \\
75 - 10x\sqrt{3} + x^2 + 200 &= 75 + 10x\sqrt{3} + x^2 \\
20x\sqrt{3} &= 200 \\
x &= \frac{10\sqrt{3}}{3}
\end{aligned}$$

We now our attention to the plane containing triangles DEF and GHI.



Since the two trapezoids below $\overline{GI}$ and $\overline{HI}$ are congruent and the two circles below the same lines are also congruent, we know that the triangle GHI posses symmetry about $\overline{FI}$.

Thus, we know that DE and GH are parallel, and we can consequently form similar triangles GHI and DEI. We can then set up the following relations.

$$\begin{aligned}
\frac{DE}{GH} &= \frac{ID}{IG} \\
\frac{DE}{GH} &= \frac{x}{x+DG} \\
\frac{DE}{10\sqrt{3}} &= \frac{\frac{10\sqrt{3}}{3}}{\frac{10\sqrt{3}}{3}+5\sqrt{3}} \\
DE &= 4\sqrt{3}
\end{aligned}$$

All that is left is to find FM. We first find the length of FI. Since we have symmetry, we know that GFI is a right triangle, and thus we can use the Pythagorean Theorem.

$$\begin{aligned}
FI &= \sqrt{GI^2 - GF^2} \\
FI &= \sqrt{\left(\frac{10\sqrt{3}}{3}+5\sqrt{3}\right)^2 - \left(5\sqrt{3}\right)^2} \\
FI &= \frac{20\sqrt{3}}{3}
\end{aligned}$$

Now we can use the similar triangles GFI and DMI.

$$\begin{aligned}
\frac{FM}{FI} &= \frac{GD}{GI} \\
\frac{FM}{FI} &= \frac{GD}{GD+x} \\
\frac{FM}{\frac{20\sqrt{3}}{3}} &= \frac{5\sqrt{3}}{5\sqrt{3}+\frac{10\sqrt{3}}{3}} \\
FM &= 4\sqrt{3}
\end{aligned}$$

Finally, the area of $DEF = \frac{1}{2}(DE)(FM) = \frac{1}{2}(4\sqrt{3})(4\sqrt{3}) = 24$

$\boxed{Area_{DEF} = 24}$

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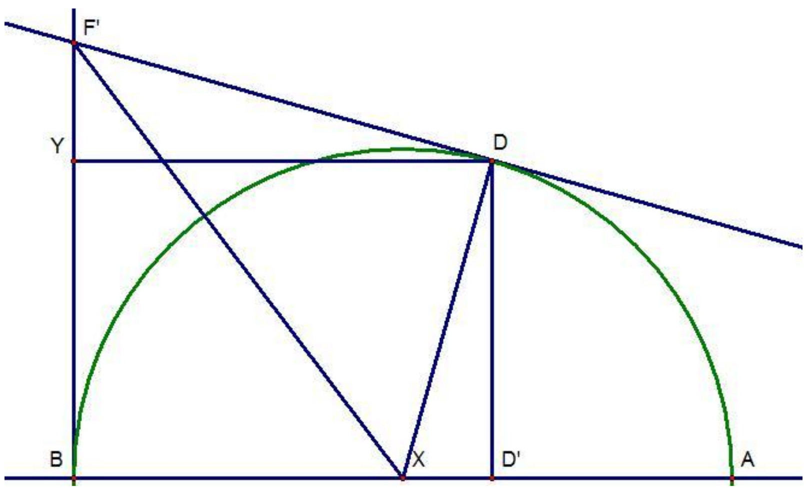
Solution 2 by: Tony Liu (10/IL)

Construct a line through F, parallel to BC, and let it intersect plane BDA at point F'. Since $BC||DE$, F' lies on the plane DEF. Now, let us focus on plane BDA and employ methods of two-dimensional geometry.



Let D' be the foot of the perpendicular from D to AB, and denote the midpoint of AB by X. Additionally, let the foot of the perpendicular from D to BF' be Y. Note that $F'D = F'B$ are tangents to the circle, since F' lies in the planes BFC and DEF. This, along with $BX = DX$ implies that $\triangle F'XB \cong \triangle F'XD$. We note that $F'X = \sqrt{BX^2 + F'B^2} = \sqrt{50 + 75} = 5\sqrt{5}$, since $F'B$ is a radius of the semicircle with diameter $BC = 10\sqrt{3}$. Letting $\theta = \angle F'XB = \angle F'XD$, we have,

$$\cos\theta = \frac{\sqrt{2}}{\sqrt{5}} \Longrightarrow \cos 2\theta = 2\cos^2\theta - 1 = -\frac{1}{5} \Longrightarrow \cos(180 - 2\theta) = \frac{1}{5}$$

and since $\angle DXD' = 180 - 2\theta$, it follows that $XD' = \sqrt{2}$, so $DD' = \sqrt{50 - 2} = 4\sqrt{3}$. In particular, we observe that $\angle BXD$ is obtuse, so D' lies on segment DA. Next, we note that $BYDD'$ is a rectangle (by construction) so $BY = DD'$ and $BD' = YD$. We will use these results later on.

Note that $\triangle DEF$ is isosceles (by symmetry of $\triangle ABC$), and since $AD' = 4\sqrt{2}$, by symmetry and similar isosceles triangles (projecting DE onto $\triangle ABC$), we deduce that $DE = 4\sqrt{3}$. Now, let h denote the altitude of $\triangle DEF$. By using the perpendicular bisector of DE (parallel to plane ABC) and a perpendicular from F to BC, we can calculate h by using the

Pythagorean Theorem. The right triangle has one leg of length $\dfrac{1-DE}{BC}\cdot\sqrt{200-75}=3\sqrt{5}$ and another of length $F'Y=\sqrt{75-72}=\sqrt{3}$. Thus, we get $h=\sqrt{45+3}=4\sqrt{3}$, and consequently the area of $\triangle DEF$ is $\frac{1}{2}\cdot h\cdot DE=\frac{1}{2}\cdot 4\sqrt{3}\cdot 4\sqrt{3}=24$.