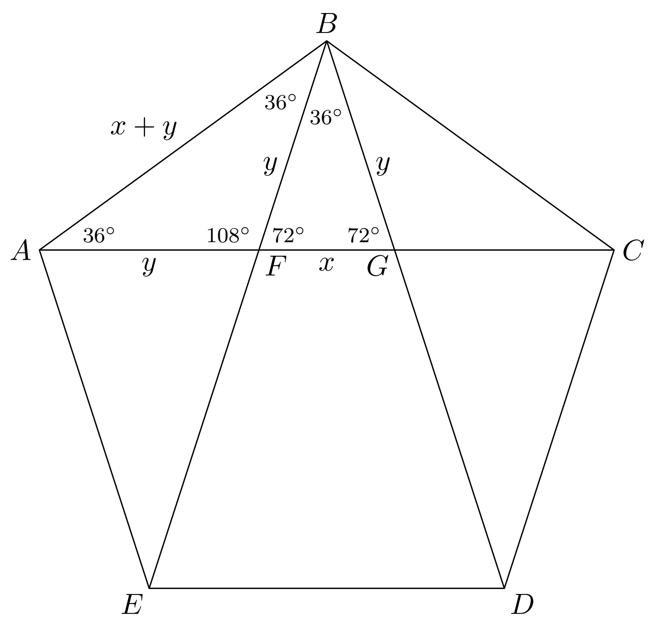
5/2/19. Faces ABC and XYZ of a regular icosahedron are parallel, with the vertices labeled such that $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ are concurrent. Let $\mathcal{S}$ be the solid with faces ABC, AYZ, BXZ, CXY, XBC, YAC, ZAB, and XYZ. If $AB = 6$, what is the volume of $\mathcal{S}$?

Comments The volume of $\mathcal{S}$ can be found by relating it to other polyhedra whose volumes are known. In the solution below, the polyhedron $\mathcal{S}$ is related to a hexagonal prism. *Solutions edited by Naoki Sato.*

Solution by: Luyi Zhang (9/CT)

First, we claim that in a regular pentagon, the ratio of the lengths of a diagonal to a side is $\frac{1+\sqrt{5}}{2} : 1$. Let $ABCDE$ be a regular pentagon. Let F be the intersection of $\overline{AC}$ and $\overline{BE}$, and let G be the intersection of $\overline{AC}$ and $\overline{BD}$.



Let $x = FG$ and $y = BG$. Triangle BFG is isosceles, so $BF = BG = y$. Triangle ABF is isosceles, so $AF = BF = y$. Then $AG = x + y$, and since triangle ABG is isosceles, $AB = x + y$.

By AAA, triangles ABG and BFG are similar, so

$$\frac{AB}{BG} = \frac{BF}{FG} \quad \Rightarrow \quad \frac{x+y}{y} = \frac{y}{x},$$

which simplifies as $y^2 - xy - x^2 = 0$. By the quadratic formula,

$$y = \frac{x \pm \sqrt{5x^2}}{2} = \frac{1 \pm \sqrt{5}}{2}\, x.$$

Since $\frac{1-\sqrt{5}}{2}$ is negative,

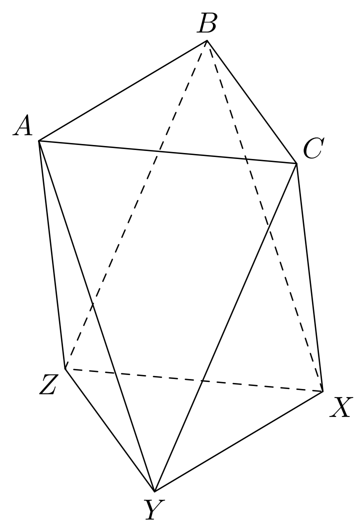
$$y = \frac{1+\sqrt{5}}{2}\, x.$$

By AAA, triangles ABC and AFB are similar, so the ratio of the lengths of a diagonal to a side is

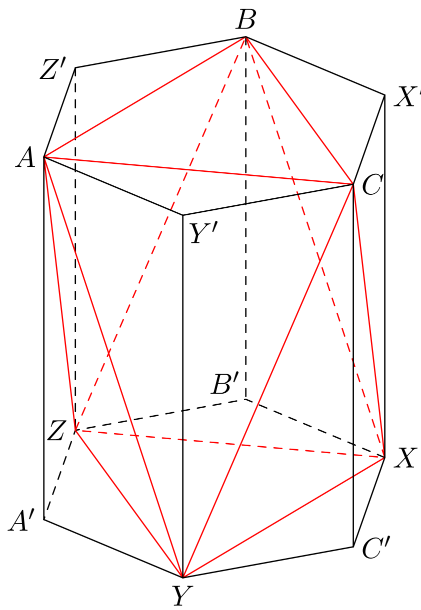
$$\frac{AC}{AB} = \frac{AB}{AF} = \frac{x+y}{y} = \frac{y}{x} = \frac{1+\sqrt{5}}{2},$$

as desired.

Now, the polyhedron $\mathcal{S}$ has eight faces. Two of them are equilateral triangles with the same side length as the icosahedron, which is 6. The other six are isosceles triangles. In these six faces, the base is the same as the side length of the icosahedron, which is 6, and the legs are the diagonals of a regular pentagon with side length 6, which is $3(1+\sqrt{5})$.



Let A', B', and C' be the projections of A, B, and C onto the plane of triangle XYZ, and let X', Y', and Z' be the projections of X, Y, and Z onto the plane of triangle ABC. Then $AY'CX'BZ'A'YC'XB'Z$ is a regular hexagonal prism. (This follows from the symmetry of the icosahedron.)



We see that this hexagonal prism is the union of polyhedron $\mathcal{S}$ (whose edges are in red) and the six tetrahedra $AA'YZ$, $BB'XZ$, $CC'XY$, $XX'BC$, $YY'AC$, and $ZZ'AB$, all of which are congruent. Thus, we can find the volume of $\mathcal{S}$ by finding the volume of the hexagonal prism, and then subtracting the volume of the six tetrahedra.

Let b denote the area of regular hexagon $AY'CX'BZ'$. Since $AB = 6$, the side length of hexagon $AY'CX'BZ'$ is $2\sqrt{3}$. Hence, the hexagon is composed of six equilateral triangles of side length $2\sqrt{3}$, so

$$b = 6 \cdot \frac{\sqrt{3}}{4}(2\sqrt{3})^2 = 18\sqrt{3}.$$

We can then use Pythagoras to find the height h of the prism. Since $\angle AA'Y = 90^\circ$, $AY^2 = (AA')^2 + (A'Y)^2$. But $AY = 3(1+\sqrt{5})$ and $A'Y = 2\sqrt{3}$, so

$$\begin{aligned}(AA')^2 &= AY^2 - (A'Y)^2 \\ &= [3(1+\sqrt{5})]^2 - (2\sqrt{3})^2 \\ &= 42 + 18\sqrt{5}.\end{aligned}$$

Therefore, $h = AA' = \sqrt{42 + 18\sqrt{5}} = \sqrt{3}(3+\sqrt{5})$.

The volume of the hexagonal prism is bh. Since triangle $A'YZ$ has $\frac{1}{6}$th the area of hexagon $AY'CX'BZ'$, the volume of tetrahedron $AA'YZ$ is

$$\frac{1}{3} \cdot \frac{1}{6} b \cdot h = \frac{1}{18} bh.$$

All six tetrahedra $AA'YZ$, $BB'XZ$, $CC'XY$, $XX'BC$, $YY'AC$, $ZZ'AB$ have the same volume, so the volume of $\mathcal{S}$ is

$$bh - \frac{6}{18}bh = bh - \frac{1}{3}bh = \frac{2}{3}bh = \frac{2}{3} \cdot 18\sqrt{3} \cdot \sqrt{3}(3+\sqrt{5}) = 108 + 36\sqrt{5}.$$