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Solutions to Problem 5/2/17
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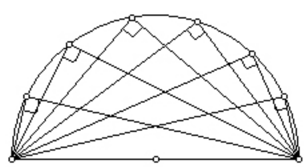
5/2/17. Given acute triangle $\triangle ABC$ in plane $\mathcal{P}$, a point Q in space is defined such that $\angle AQB = \angle BQC = \angle CQA = 90°$. Point X is the point in plane $\mathcal{P}$ such that $\overline{QX}$ is perpendicular to plane $\mathcal{P}$. Given $\angle ABC = 40°$ and $\angle ACB = 75°$, find $\angle AXC$.

Credit This problem was proposed by Sandor Lehoczky.

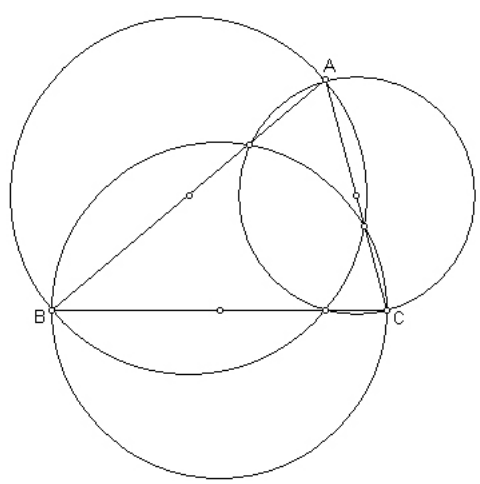
Comments The key insight in this problem is to realize that X is the orthocenter of triangle ABC. This, in turn, can be proven by showing that Q must lie on certain spheres. Once you identify that the point X is the orthocenter, the rest of the problem becomes an easy angle chase. *Solutions edited by Naoki Sato.*

Solution 1 by: Philip Shirey (12/PA)

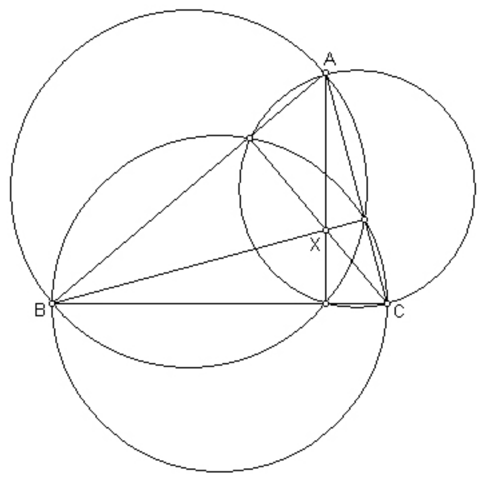
It is well known that in two dimensions, given points A and B, the locus of points P such that $\angle APB = 90°$ is the circle with diameter AB. (See diagram below.) The three-dimensional locus, such as in this problem, is a sphere. Thus, point Q is the intersection of three spheres.



We can better visualize the intersection of the three spheres by taking their intersection with the plane $\mathcal{P}$.



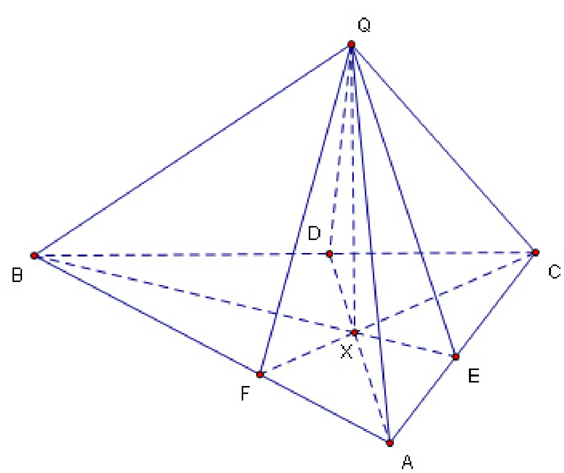
The triangle shown above is acute, so each circle interesects with the adjancent segments. As mentioned in the above theorem, any point on a circle will form a right triangle with its diameter. Because of this, these intersections form the altitudes of the triangle.



The intersection of two spheres is a circle. In the 2D representation of our 3D figure, these circles would be perpendicular to the plane of the page, which means they would be shown as segments in the 2D image. The endpoints of these segments are the interesections of the projected spheres (i.e., the circles). So, the altitudes represent the intersections of the spheres. As point X lies on plane $\mathcal{P}$, X is the intersection of the altitudes, otherwise known as the orthocenter.

Lastly, $\angle AXC = 140^\circ$ because the congruent angle on the opposite side of X forms a quadrilateral with two 90° angles with $\angle ABC$, which is 40°. So $\angle AXC = 180^\circ - 40^\circ = 140^\circ$.

Solution 2 by: Tan Zou (10/IN)



Let P_{ACQ} be the plane of $\triangle ACQ$. Draw a line through B and X such that it intersects side $\overline{AC}$ at point E. Draw EQ and let P_{BEQ} be the plane of $\triangle BEQ$. Then $\overline{BQ} \perp \overline{AQ}$ and

$\overline{BQ} \perp \overline{CQ}$, so $P_{BEQ} \perp P_{ACQ}$. Hence, $\overline{BE} \perp \overline{AC}$, because $\overline{BE}$ and $\overline{AC}$ are in P_{BEQ} and P_{ACQ}, respectively.

Similarly, if we draw $\overline{AD}$ and $\overline{CF}$ through point X, we can prove that they are perpendicular to $\overline{BC}$ and $\overline{AB}$, respectively. Therefore, X is the orthocenter of $\triangle ABC$.

We know that if H is the orthocenter of $\triangle ABC$, then $\angle A + \angle BHC = 180°$. To see this, let $\angle A = \theta$. Then $\angle ABH = 90° - \theta$, so $\angle BHF = \theta$, and $\angle BHC = 180° - \theta$.

Since X is the orthocenter and $\angle B = 40°$, $\angle AXC = 180° - 40° = 140°$.

Solution 3 by: Zhou Fan (12/NJ)

Let X be the origin of coordinate space, and let us use the notation $\vec{a}$ for the vector from point X to point A, etc.

Since QX is perpendicular to the plane containing ABC,

$$\vec{q} \cdot \vec{a} = \vec{q} \cdot \vec{b} = \vec{q} \cdot \vec{c} = 0.$$

Since $\angle AQB = \angle BQC = \angle CQA = 90°$,

$$(\vec{q} - \vec{a}) \cdot (\vec{q} - \vec{b}) = (\vec{q} - \vec{b}) \cdot (\vec{q} - \vec{c}) = (\vec{q} - \vec{c}) \cdot (\vec{q} - \vec{a}) = 0.$$

Expanding (by the distributive property of dot products), and substituting 0 for $\vec{q} \cdot \vec{a}$, $\vec{q} \cdot \vec{b}$, and $\vec{q} \cdot \vec{c}$ gives $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = -|\vec{q}|^2$. But if $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, then $\vec{a} \cdot (\vec{b} - \vec{c}) = 0$, so AX is perpendicular to BC. Similarly, BX and CX are perpendicular to AC and AB, respectively, so X is the orthocenter of ABC.

Let CX intersect AB at C_1 and AX intersect BC at A_1; then $\triangle AC_1X \cong \triangle AA_1B$ because $\angle AC_1X = \angle AA_1B = 90°$. Therefore, $\angle AXC_1 = \angle ABA_1 = 40°$, and $\angle AXC = 140°$.