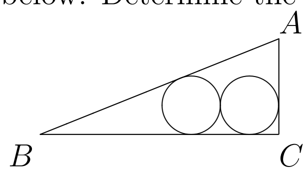
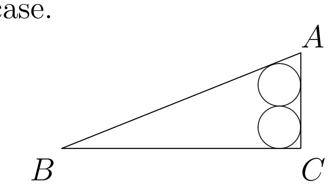
5/2/16.

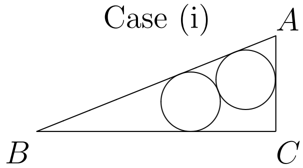
Two circles of equal radius can tightly fit inside right triangle ABC, which has $AB = 13$, $BC = 12$, and $CA = 5$, in the three positions illustrated below. Determine the radii of the circles in each case.



Case (i)



Case (ii)



Case (iii)

Credit This problem was inspired by Problem 5.3.2 in *Traditional Japanese Mathematics Problems of the 18th and 19th Centuries* published in 2002 by SCT Publishing of Singapore. We are thankful to Mr. Willie Yong, the Publisher, for sending us a copy of this wonderful book.

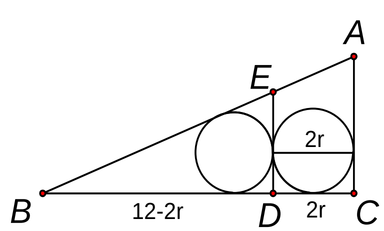
Comments There are many ways to solve this problem though most of them used right angles drawn from the centers of the circles to points of tangency and the use of similar triangles (whether or not they were implied by the use of trigonometry).

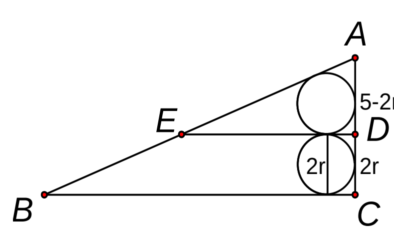
Solution 1 by: Jeffrey Manning (9/CA)

The method for solving case (iii) is different than the method for solving cases (i) and (ii) so we will look at these seperatly.

Cases (i) and (ii)

Draw the line through the point of intersection of the two circles that is tangent to both circles. Call the point where this line intersects the hypotenuse, E, and call the point where it intersects the leg that is tangent to both circles (BC in case (i) and CA in case (ii)), D.





In each case ED is perpendicular to one leg and parallel to the other. This means that in case (i) $\triangle ABC \sim \triangle EBD$ and in case (ii) $\triangle ABC \sim \triangle AED$. In each of these cases one of the circles is the incircle of the new triangle. Also because in each case the distance from ED to the leg it is parallel to is $2r$ (where r is the length of the radii of the circles) we have $DC = 2r$

The length of the radius, R, of the incircle of a right triangle is given by $R = \dfrac{ab}{a+b+c}$. Since $a = 5$, $b = 12$ and $c = 13$, we have $R = 2$. R will be proportional to r.

We will now solve each case seperatly.

Case (i)

Since $DC = 2r$ we have $BD = 12 - 2r$ so, since $\triangle ABC \sim \triangle EBD$,

$$\begin{aligned} \frac{BD}{BC} &= \frac{r}{R} \\ \frac{12-2r}{12} &= \frac{r}{2} \\ 24 - 4r &= 12r \\ 16r &= 24 \\ \mathbf{r} &= \mathbf{3/2} \end{aligned}$$

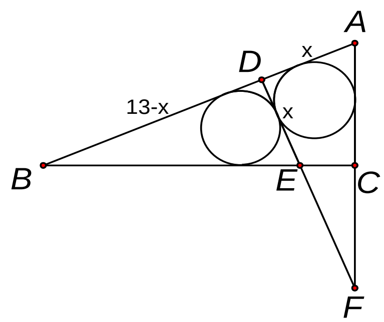
Case (ii)

Since $DC = 2r$ we have $AD = 5 - 2r$ so, since $\triangle ABC \sim \triangle EBD$,

$$\begin{aligned} \frac{AD}{AC} &= \frac{r}{R} \\ \frac{5-2r}{5} &= \frac{r}{2} \\ 10 - 4r &= 5r \\ 9r &= 10 \\ \mathbf{r} &= \mathbf{10/9} \end{aligned}$$

Case (iii)

Draw the line tangent to both circles similarly to cases (i) and (ii). Call the points where it intersects AB, BC and the line formed by AC, D, E and F respectively.



$\triangle EBD$ and $\triangle AFD$ are both right triangles because $AB \perp DF$. Also $\angle ABC = \angle EBD$ and $\angle BAC = \angle FAD$, so $\triangle EBD \sim \triangle ABC \sim \triangle AFD$. Since $\triangle EBD \sim \triangle AFD$ and the radii of their incircles are equal we have $\triangle EBD \cong \triangle AFD$, so $AD = DE$.

Let $x = AD = DE$

Since $AD = x$, $DB = 13 - x$ which gives:

$$\frac{x}{CA} = \frac{13-x}{BC}$$

$$\begin{aligned} 12x &= 65 - 5x \\ 17x &= 65 \\ x &= 65/17 \end{aligned}$$

$$\begin{aligned} \frac{r}{R} &= \frac{x}{CA} \\ \frac{r}{2} &= \frac{x}{5} \\ r &= \frac{2}{5}x \\ r &= \left(\frac{2}{5}\right)\left(\frac{65}{17}\right) \\ \mathbf{r} &= \mathbf{26/17} \end{aligned}$$

This can be generalized for $AB = c$, $BC = a$ and $CA = b$:

Case (i)

We have $BD = a - 2r$ so

$$\mathbf{r} = \frac{\mathbf{aR}}{\mathbf{a+2r}}$$

Substituting $\dfrac{ab}{a+b+c}$ for R gives:

$$\mathbf{r} = \frac{\mathbf{ab}}{\mathbf{a+3b+c}}$$

Similarly in case (ii) $\mathbf{r} = \dfrac{\mathbf{ab}}{\mathbf{3a+b+c}}$

Case (iii)

Since $AD = x$ we have $DB = c - x$ so

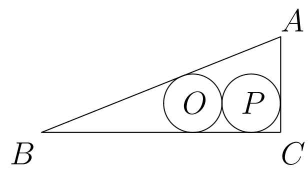
$$\begin{aligned} \frac{x}{b} &= \frac{c-x}{a} \\ x &= \frac{bc}{a+b} \end{aligned}$$

So

$$r = \frac{R}{b}x = \frac{Rc}{a+b}$$

$$\mathbf{r} = \frac{\mathbf{abc}}{(\mathbf{a+b+c})(\mathbf{a+b})}$$

Solution 2 by: Chenyu Feng (12/IL)
Case (i)

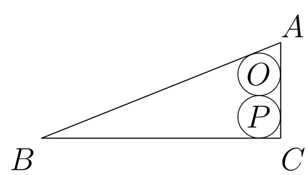


Call the centers of the two circles O and P, with P being closer to point C. Because the area of the triangle is $5 \cdot 12/2 = 30$, we will sum the partial areas to find r.

$$\begin{aligned} K(\text{trapezoid } BCPO) + K(\triangle AOB) + K(\triangle AOP) + K(\triangle APC) &= K(\triangle ABC) \\ \frac{r}{2} \cdot (2r+12) + \frac{13r}{2} + \frac{2r(5-r)}{2} + \frac{5r}{2} &= 30 \\ r &= 3/2 \end{aligned}$$

Thus the radii of both circles in case (i) is $\boxed{3/2}$.

Case (ii)



Call the centers of the two circles O and P, with P being closer to point C. Because the area of the triangle is $5 \cdot 12/2 = 30$, we will again sum the partial areas to find r.

$$\begin{aligned} K(\text{trapezoid } AOPC) + K(\triangle BOA) + K(\triangle BOP) + K(\triangle CPB) &= K(\triangle ABC) \\ \frac{r}{2} \cdot (2r+5) + \frac{13r}{2} + \frac{2r(12-r)}{2} + \frac{12r}{2} &= 30 \\ r &= 10/9 \end{aligned}$$

Thus the radii of both circles in case (ii) is $\boxed{10/9}$.

Case (iii)

We will use similar triangles to solve this one. Firstly, draw the two circles' common internal tangent, intersecting line AB at D and line BC at E. Extend lines DE and AC so that they meet at point F. Because $\angle BDE = \angle ACE$ (they're both right angles), $\angle DEB = \angle A$ and the two incircles are the same size, $\triangle ADF \cong \triangle BDE$. It immediately follows that $AD = DE$. Because $\triangle BDE$ and $\triangle ABC$ each have a right angle and they share $\angle B$, it follows that $\triangle BDE \sim \triangle ABC$. Then, we set up a ratio:

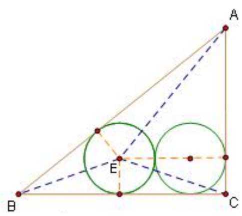
$$\begin{aligned}
\frac{DE}{AC} &= \frac{BD}{BC} \\
\frac{DE}{5} &= \frac{13-DE}{12} \\
DE &= \frac{65}{17} \\
BD &= 13 - DA = 13 - DE = \frac{156}{17} \\
BE &= \sqrt{DE^2 + BD^2} = \sqrt{\left(\frac{65}{17}\right)^2 + \left(\frac{156}{17}\right)^2} = \frac{169}{17}
\end{aligned}$$

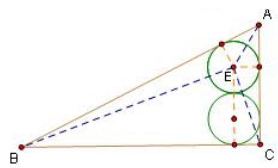
And since in a right triangle, $2r = a + b - c$, where a and b are the lengths of the legs and c is the length of the hypotenuse:

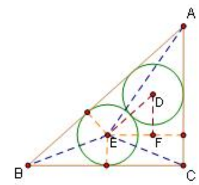
$$\begin{aligned}
2r &= a + b - c \\
r &= \frac{a+b-c}{2} \\
r &= \frac{\frac{156}{17} + \frac{65}{17} - \frac{169}{17}}{2} \\
r &= \frac{26}{17}
\end{aligned}$$

And thus the radius of the circles, r, is $\boxed{26/17}$.

Solution 3 by: Benjamin Lee (9/MD)

Case (i) 

Case (ii) 

Case (iii) 

In all cases, $\text{Area}_{\text{3triangles}} = \text{Area}_{\triangle ABC} = \frac{1}{2}(BC)(AC) = \frac{1}{2}(12)(5) = 30$

In all cases, we use the formula for a triangle area for the 3 triangles ($\triangle EBA$, $\triangle EAC$, $\triangle ECB$) and $\triangle ABC$, and r, b, h, for the radius, base, and height of the small triangle, respectively, so we have

$$\frac{1}{2}BC * h_{BC} + \frac{1}{2}AC * h_{AC} + \frac{1}{2}AB * h_{AB} = 30$$

$$6h_{BC} + \frac{5}{2}h_{AC} + \frac{13}{2}h_{AB} = 30.$$

Case (i)

Drawing auxillary lines (orange) from the center of the leftmost circle to the vertices, we have three small triangles whose area add up to the area of $\triangle ABC$. Thus,

Using the values $h_{BC} = r$, $h_{AC} = 3r$, $h_{AB} = r$, we get

$$6r + \frac{5}{2}(3r) + \frac{13}{2}r = 30$$

$$20r = 30$$

$$r = \frac{3}{2}$$

Case (ii)

Drawing auxillary lines (orange) from the center of the topmost circle to the vertices, we have three small triangles whose area add up to the area of $\triangle ABC$. Thus,

Using the values $h_{BC} = 3r$, $h_{AC} = r$, $h_{AB} = r$, we get
$6(3r) + \frac{5}{2}r + \frac{13}{2}r = 30$
$27r = 30$
$r = \frac{10}{9}$

Case (iii)

Drawing auxillary lines (orange) from the center of the leftmost circle to the vertices, we have three small triangles whose area add up to the area of $\triangle ABC$.

To find the distance from the center of the leftmost circle to AC, because $\triangle ABC \sim \triangle DEF$ we can use the proportion

$$\frac{h_{AC} - r}{BC} = \frac{DE}{AB} = \frac{2r}{AB}.$$

From this proportion,

$$h_{AC} = \frac{24}{13}r + r.$$

Using the values $h_{BC} = r$, $h_{AC} = \frac{24}{13}r + r$, $h_{AB} = r$, we get

$$6r + \frac{13}{2}r + \frac{5}{2}(r + \frac{24}{13}r) = 30$$

$$r = \frac{26}{17}$$

Thus the radii of the circles of Cases (i),(ii), and (iii), are $\frac{3}{2}$, $\frac{10}{9}$, and $\frac{26}{17}$, respectively.