USA Mathematical Talent Search

Solutions to Problem 5/1/19

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5/1/19. Let c be a real number. The sequence a_1, a_2, a_3, $\ldots$ is defined by $a_1 = c$ and $a_n = 2a_{n-1}^2 - 1$ for all $n \geq 2$. Find all values of c such that $a_n < 0$ for all $n \geq 1$.

Credit This problem was proposed by Naoki Sato.

Comments It is not difficult to show that the value $c = -\frac{1}{2}$ works. If $c \neq -\frac{1}{2}$, then the terms of the sequence must diverge from $-\frac{1}{2}$, to the point where they become positive. The following solution uses a rigorous bounding argument. *Solutions edited by Naoki Sato.*

Solution 1 by: Sam Elder (12/CO)

The only value is $c = -\frac{1}{2}$.

If $a_n = -\frac{1}{2}$, then $a_{n+1} = 2a_n^2 - 1 = 2(-\frac{1}{2})^2 - 1 = -\frac{1}{2}$, so if $c = -\frac{1}{2}$, then $a_n = -\frac{1}{2} < 0$ for all $n \geq 1$ and the result is achieved.

Assume $c \neq -\frac{1}{2}$, and define the sequence $b_n = 2a_n + 1$. Assume that $a_n < 0$ for all n, so $b_n < 1$ for all n. A recursion for the b_n is derived from that for the a_n:

$$\begin{aligned}
\frac{b_n - 1}{2} &= 2\left(\frac{b_{n-1} - 1}{2}\right)^2 - 1 \\
\Rightarrow \quad b_n - 1 &= (b_{n-1} - 1)^2 - 2 \\
\Rightarrow \quad b_n &= b_{n-1}(b_{n-1} - 2) \\
\Rightarrow \quad b_n &= b_{n-2}(2 - b_{n-2})(2 - b_{n-1})
\end{aligned}$$

for all $n > 2$. If $b_n = 0$, then $b_{n-1} = 0$ or $b_{n-1} = 2$. However, by assumption, $b_n < 1$ for all n, and $b_1 = 2a_1 + 1 = 2c + 1 \neq 0$, so $b_n \neq 0$ for all n.

If $b_n < 0$, then $b_{n+1} = b_n(b_n - 2) > 0$. Likewise, if $b_n > 0$, then $b_{n+1} < 0$ since $b_n < 1$ and so $b_n - 2 < -1 < 0$. Hence, the terms b_n alternate in sign, so for all n, one of b_{n-1} and b_{n-2} is negative. The other is less than 1, so

$$\frac{b_n}{b_{n-2}} = (2 - b_{n-2})(2 - b_{n-1}) > (2 - 0)(2 - 1) = 2.$$

Let $m = 1$ if b_1 is positive, and $m = 2$ if b_2 is positive, so b_m is positive. Take l sufficiedntly large so that $b_m > 2^{-l}$. Then

$$b_{m+2l} > 2b_{m+2(l-1)} > 2^2 b_{m+2(l-2)} > \cdots > 2^{l-1} b_{m+2} > 2^l b_m > 1,$$

a contradiction. So $c = -\frac{1}{2}$ is the only solution.