4/4/19. Suppose that w, x, y, z are positive real numbers such that $w + x < y + z$. Prove that it is impossible to simultaneously satisfy both

$$(w + x)yz < wx(y + z) \qquad \text{and} \qquad (w + x)(y + z) < wx + yz.$$

Comments Since we want to show that not all three inequalities can hold simultaneously, we can approach the problem by using contradiction. *Solutions edited by Naoki Sato.*

Solution 1 by: Andy Zhu (11/NJ)

For the sake of contradiction, suppose that all the given inequalities hold. Multiplying the inequalities $wx(y + z) > (w + x)yz$ and $wx + yz > (w + x)(y + z)$, we get

$$wx(y + z)(wx + yz) > yz(w + x)^2(y + z).$$

By the AM-GM inequality, $(w + x)^2 \geq 4wx$, so

$$wx(y + z)(wx + yz) > yz(w + x)^2(y + z) \geq 4wxyz(y + z).$$

Dividing by $wx(y + z)$ (which is positive), we get

$$wx + yz > 4yz,$$

so $wx > 3yz$.

Also, since $y + z > w + x$ and $wx + yz > (w + x)(y + z)$,

$$wx + yz > (w + x)(y + z) > (w + x)^2 \geq 4wx,$$

so $yz > 3wx$. Multiplying the inequalities $wx > 3yz$ and $yz > 3wx$, we get $wxyz > 9wxyz$, contradiction. Thus, not all the given inequalities can hold simultaneously.

Solution 2 by: Kristin Cordwell (11/NM)

We argue by contradiction. Suppose that the positive real numbers w, x, y, z satisfy all the given inequalities, so $w + x < y + z$,

$$(w + x)yz < wx(y + z) \quad \Rightarrow \quad wxy + wxz - wyz - xyz > 0,$$

and

$$(w + x)(y + z) < wx + yz \quad \Rightarrow \quad wx + yz - wy - xy - wz - xz > 0.$$

Now consider the polynomial $p(s) = (s - w)(s - x)(s + y)(s + z)$. Expanding this, we have

$$\begin{aligned} p(s) = s^4 &+ (y + z - w - x)s^3 + (wx + yz - wy - xy - wz - xz)s^2 \\ &+ (wxy + wxz - wyz - xyz)s + wxyz. \end{aligned}$$

The coefficients of s^3, s^2, and s are all positive, and $wxyz > 0$ because w, x, y, $z > 0$. Therefore, $p(s) > 0$ for all $s > 0$. However, $p(w) = 0$ and $w > 0$, contradiction. Therefore, all three inequalities cannot simultaneously hold.