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Solutions to Problem 4/4/17
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4/4/17. Find, with proof, all irrational numbers x such that both $x^3 - 6x$ and $x^4 - 8x^2$ are rational.

Credit This problem was proposed by Erin Schram.

Comments Many students were able to find the solutions $\pm\sqrt{6}$, but there are four additional solutions. One must be careful about making assumptions about irrational numbers, such as what they must look like. The best strategy here is to play the two given polynomials off of each other. *Solutions edited by Naoki Sato.*

Solution 1 by: Tony Liu (11/IL)

We claim that $x = \pm 1 \pm \sqrt{3}$ (taking all four combinations of signs) and $x = \pm\sqrt{6}$ are the only six irrational x such that both $x^3 - 6x$ and $x^4 - 8x^2$ are rational. Now, we prove that these are the only such values.

Assume we have some irrational x such that both $x^3 - 6x$ and $x^4 - 8x^2$ are rational. Let $a = x^2 - 4$, so $a^2 = x^4 - 8x^2 + 16$ is rational. Let $b = x^3 - 6x = x(x^2 - 6) = x(a - 2)$ which is also rational by hypothesis. We have

$$b^2 = x^2(a-2)^2 = (a+4)(a-2)^2 = a^3 - 12a + 16 = a(a^2 - 12) + 16.$$

In particular, because b^2 is rational, $a(a^2 - 12)$ must be rational. If $a^2 - 12 \neq 0$, then $a^2 - 12$ is rational so a must be rational as well. Otherwise, $a = \pm 2\sqrt{3}$.

If $a^2 - 12 \neq 0$ and a is rational, note that $b = x(a-2)$ is rational. Because x is irrational (and $x \neq 0$) we must have $a = 2$. Thus, $x^2 = 6$ and $x = \pm\sqrt{6}$.

If $a = \pm 2\sqrt{3}$ then $x^2 = 4 \pm 2\sqrt{3} = (1 \pm \sqrt{3})^2$ so $x = \pm 1 \pm \sqrt{3}$ (taking all four combinations of signs). It is easily verified that all six solutions make $x^3 - 6x$ and $x^4 - 8x^2$ rational and this concludes our proof.