
4/3/17. Find, with proof, all triples of real numbers (a, b, c) such that all four roots of the polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + b$ are positive integers. (The four roots need not be distinct.)

Credit This problem was based on a proposal by Brian Rice.

Comments To find all possible sets of roots (which is what the problem is effectively asking for), you must use both the fact they are positive and integers. The first condition can be used to find bounds on the roots and narrow down to a finite number of cases, and the second condition can be used to find them specifically. *Solutions edited by Naoki Sato.*

Solution 1 by: Tony Liu (11/IL)

Let p, q, r, and s be the positive integer roots of $f(x) = x^3 + ax^3 + bx^2 + cx + b$. We have

$$\begin{aligned} f(x) &= x^4 + ax^3 + bx^2 + cx + b \\ &= (x-p)(x-q)(x-r)(x-s) \\ &= x^4 - (p+q+r+s)x^3 + (pq+pr+ps+qr+qs+rs)x^2 \\ &\quad - (pqr + qrs + rsp + spq)x + pqrs. \end{aligned}$$

Comparing coefficients, we note that it suffices to find all quadruples (p, q, r, s) of positive integers such that

$$b = pqrs = pq + pr + ps + qr + qs + rs,$$

for then we obtain reals (which in fact are integers) a and c as

$$a = -(p+q+r+s) \quad \text{and} \quad c = -(pqr + qrs + rsp + spq),$$

and hence obtain all triples (a, b, c). First, let us rewrite our equation containing b by dividng through by $pqrs$. We have

$$\frac{1}{pq} + \frac{1}{pr} + \frac{1}{ps} + \frac{1}{qr} + \frac{1}{qs} + \frac{1}{rs} = 1.$$

Without loss of generality, we have $\frac{1}{rs} \geq \frac{1}{6}$ so $rs \leq 6$ for some pair of the roots (for instance, the two smallest). Assume $r \geq s$ and $p \geq q$. We now proceed with some casework.

Case 1: $rs = 6$. We either have $(r, s) = (6, 1)$ or $(3, 2)$. If $(r, s) = (6, 1)$, we have

$$6pq = pq + 7(p+q) + 6 \iff (5p-7)(5q-7) = 49 + 30 = 79,$$

which admittedly does not have any integer solutions (p, q) since 79 is prime and we must have $5p - 7 = 79$ and $5q - 7 = 1$, but this is clearly impossible. If $(r, s) = (3, 2)$ we have

$$6pq = pq + 5(p+q) + 6 \iff 5(pq - p - q) = 6,$$

which again does not have any integer solutions (p, q) since 6 is not divisible by 5.

Case 2: $rs = 5$. We have $(r, s) = (5, 1)$ so

$$5pq = pq + 6(p + q) + 5 \iff 2(pq - 3p - 3q) = 5,$$

which does not have any integer solutions (p, q) since 5 is odd.

Case 3: $rs = 4$. We either have $(r, s) = (4, 1)$ or $(2, 2)$. If $(r, s) = (4, 1)$, then

$$4pq = pq + 5(p + q) + 4 \iff (3p - 5)(3q - 5) = 25 + 12 = 37,$$

so $3p - 5 = 37 \Rightarrow p = 14$ and $3q - 5 = 1 \Rightarrow q = 2$, since 37 is prime. Thus, we obtain $(p, q, r, s) = (14, 2, 4, 1)$, whence $a = -21$, $b = 112$, and $c = -204$. If $(r, s) = (2, 2)$, we have

$$4pq = pq + 4(p + q) + 4 \iff (3p - 4)(3q - 4) = 16 + 12 = 28,$$

which decomposes as a product of two positive integers as $28 \cdot 1 = 14 \cdot 2 = 7 \cdot 4$. It is easily verified that only the cases $3p - 4 = 14$ and $3q - 4 = 2$ yields a valid solution $(p, q) = (6, 2)$. We obtain $(p, q, r, s) = (6, 2, 2, 2)$, whence $a = -12$, $b = 48$, and $c = -80$.

Case 4: $rs = 3$. We have $(r, s) = (3, 1)$ so

$$3pq = pq + 4(p + q) + 3 \iff 2(pq - 2p - 2q) = 3,$$

which does not have any integer solutions (p, q) since 3 is odd.

Case 5: $rs = 2$. We have $(r, s) = (2, 1)$ so

$$2pq = pq + 3(p + q) + 2 \iff (p - 3)(q - 3) = 9 + 2 = 11,$$

and we have $p - 3 = 11 \Rightarrow p = 14$ and $q - 3 = 1 \Rightarrow q = 4$. Thus, we get $(p, q, r, s) = (14, 4, 2, 1)$, which we obtained earlier in a different order with $(r, s) = (4, 1)$.

Case 6: $rs = 1$. We have $(r, s) = (1, 1)$ so

$$pq = pq + 2(p + q) + 1 \iff 2(p + q) + 1 = 0,$$

which is clearly absurd, so there are no positive integer solutions (p, q).

Thus, we have determined all desired triples (a, b, c), namely $(-21, 112, -204)$ and $(-12, 48, -80)$.

Note: The number of cases can be reduced by the following argument. First, not all of p, q, r, and s can be odd. If they were, then $pqrs$ would be odd, but then $pq+pr+ps+qr+qs+rs$, as the sum of six odd integers, would be even. Hence, at least one of them must be even.

WOLOG, let p be even. Then

$$pqrs - pq - pr - ps = p(qrs - q - r - s) = qr + qs + rs,$$

so $qr+qs+rs$ is even. If q, r, and s were all odd, then $qr+qs+rs$ would be odd, contradiction, so at least one of them must be even. WOLOG, let q be even.

Then

$$rs = pqrs - pq - pr - ps - qr - qs.$$

Each term in the RHS contains a factor of p or q, so the RHS is even. Then rs is even, so one of r and s must be even. Hence, of the four positive integers p, q, r, and s, at least three must be even. This argument, among other things, allows us to eliminate cases 2, 4, and 6 above.