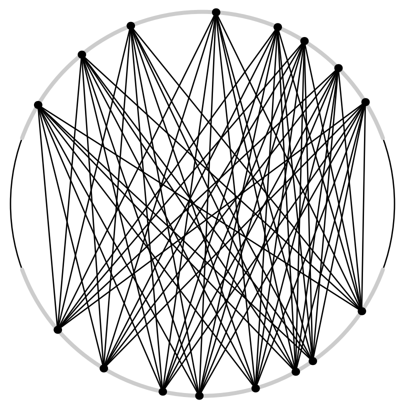
4/2/19. Two nonoverlapping arcs of a circle are chosen. Eight distinct points are then chosen on each arc. All 64 segments connecting a chosen point on one arc to a chosen point on the other arc are drawn. How many triangles are formed that have at least one of the 16 points as a vertex?

A sample figure is shown below:



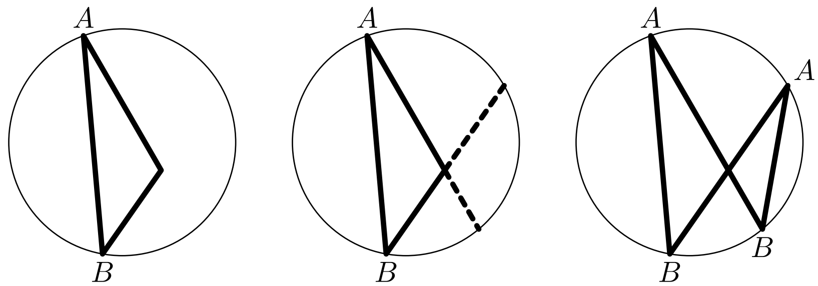
Comments We can count the number of triangles by counting the number of ways to choose points on the circle that give rise to these triangles. However, for each case, we must be careful to factor in just how many triangles are created by the points we choose. *Solutions edited by Naoki Sato.*

Solution by: Santhosh Karnik (11/GA)

To generalize, suppose m points are chosen on one arc and n points are chosen on the other arc. Let the arc with m points be arc A and the arc with n points be arc B. Since all triangles formed must have at least one vertex on the circle, a triangle must have either 1, 2, or 3 vertices on the circle.

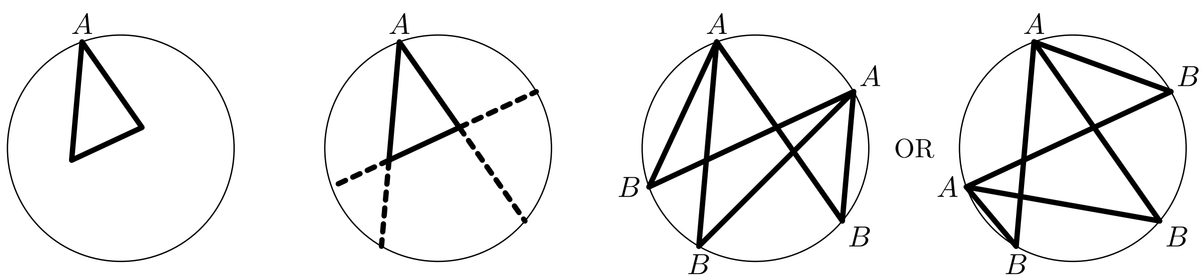
If all 3 vertices of a triangle are on the circle, then at least 2 of the 3 vertices must be on the same arc. But none of the segments connect 2 points on the same arc. Thus there are no triangles formed with all 3 vertices on the circle.

If 2 vertices of a triangle are on the circle, then one must be on the arc A, and the other must be on arc B. The third vertex can be anywhere inside the circle as long as it is connected to the other two vertices. Since all segments connect a point on arc A to a point on arc B, the two segments of the triangle that are connected to the third vertex can each be extended to a point on the circle. This gives a total of 4 points, 2 on each arc. Connecting all possible segments on these 4 points yields 2 triangles with 2 vertices on the circle. Thus, selecting 2 points on arc A and 2 points on arc B always yields 2 triangles with exactly two vertices on the circle, and all such triangles can be formed by choosing 2 points on each arc and connecting all possible segments.



Therefore, there are $2\binom{m}{2}\binom{n}{2}$ triangles formed such that 2 vertices of a triangle are on the circle.

If a triangle has only 1 vertex on the circle, then the other 2 vertices must be inside the circle and connected to each other and the first vertex. By extending all the segments of the triangle to points on the circle, 4 additional points are obtained, 1 on the same arc as the first vertex, and 3 on the other arc. Connecting all possible segments on these 5 points yields 2 triangles with exactly one vertex on the circle. Thus, selecting 2 points on arc A and 3 points on arc B or vice-versa always yields 2 triangles with 1 vertex on the circle, and all such triangles can be formed by choosing 2 points on one arc and 3 points on the other arc and connecting all possible segments.



Therefore there are $2\binom{m}{2}\binom{n}{3} + 2\binom{m}{3}\binom{n}{2}$ triangles formed such that 1 vertex of the triangle is on the circle. Therefore, the total number of triangles formed with at least one vertex on the circle is

$$T(m,n) = 2\left[\binom{m}{2}\binom{n}{2} + \binom{m}{2}\binom{n}{3} + \binom{m}{3}\binom{n}{2}\right].$$

In the problem, there are $m = n = 8$ points on each arc. So the total number of triangles formed with at least one vertex on the circle is

$$T(8,8) = 2\left[\binom{8}{2}\binom{8}{2} + \binom{8}{2}\binom{8}{3} + \binom{8}{3}\binom{8}{2}\right] = 7840.$$