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Solutions to Problem 4/2/18
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4/2/18. For every integer $k \geq 2$, find a formula (in terms of k) for the smallest positive integer n that has the following property:

> No matter how the elements of $\{1, 2, \ldots, n\}$ are colored red and blue, we can find k elements a_1, a_2, ..., a_k, where the a_i are not necessarily distinct elements, and an element b such that:
>
> (a) $a_1 + a_2 + \cdots + a_k = b$, and
>
> (b) all of the a_i's and b are the same color.

Credit This problem was proposed by Dave Patrick, and is a generalization of a problem that appeared on the 2004 British Mathematical Olympiad.

Comments There are two parts to this problem: You must show that for $n = k^2 + k - 2$, there is a coloring that does not satisfy the given property, and you must show that for $n = k^2 + k - 1$, any coloring satisfies the given property.

The first part can be accomplished by explicitly constructing a counter-example, and the second part can be shown by considering the colors of only a few key numbers. *Solutions edited by Naoki Sato.*

Solution 1 by: Sam Elder (11/CO)

The answer is $n = k^2 + k - 1$.

First, we show that for $n = k^2 + k - 2$, we can produce a coloring that does not satisfy these criteria. Let the numbers 1 to $k - 1$ be red, k to $k^2 - 1$ be blue, and k^2 to $k^2 + k - 2$ be red. Any k blue numbers sum to at least k^2, and all numbers at least k^2 are red. Also, if we choose k red numbers less than k, we get a total sum of at most $k(k - 1) < k^2$ but at least k, and all of these numbers are blue. Moreover, if we choose at least one red number that is at least k^2, our sum is at least $k^2 + k - 1$, which is not in our set. So no matter which k identically-colored numbers we choose, their sum is not the same color.

Now, we show that $n = k^2 + k - 1$ does work. Assume for the sake of contradiction that we cannot find $k + 1$ such integers as described in the problem. Without loss of generality, let 1 be red. Then k must be blue and k^2 must be red. Summing $k^2 + \underbrace{1 + \cdots + 1}_{k-1}$, $k^2 + k - 1$ must also be blue. Now this means $k + 1$ must be red, because otherwise we would have $k + \underbrace{(k + 1) + \cdots + (k + 1)}_{k-1} = k^2 + k - 1$, with k, $k + 1$ and $k^2 + k - 1$ blue, contradiction. But then we get $1 + \underbrace{(k + 1) + \cdots + (k + 1)}_{k-1} = k^2$, and 1, $k + 1$ and k^2 are red, contradiction. Therefore, for $n = k^2 + k - 1$, any coloring satisfies the given property.