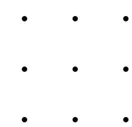
4/2/16. How many quadrilaterals in the plane have four of the nine points $(0,0)$, $(0,1)$, $(0,2)$, $(1,0)$, $(1,1)$, $(1,2)$, $(2,0)$, $(2,1)$, $(2,2)$ as vertices? Do count both concave and convex quadrilaterals, but do not count figures where two sides cross each other or where a vertex angle is 180°. Rigorously verify that no quadrilateral was skipped or counted more than once.



Credit This problem was proposed by Professor Harold Reiter, the president of Mu Alpha Theta Mathematics Honor Society and a long-time supporter of the USAMTS program.

Comments Though a combinatorial argument is the easiest place to start with this problem, many students found interesting ways to show that they covered all possible cases. Many of those methods failed due to the difficulty of making sure every case was covered. Most such methods would scale poorly to problems with larger grids.

Solution 1 by: Derrick Sund (11/NC)
There are $\binom{9}{4} = 126$ different ways to choose four of the points to be the vertices of a quadrilateral. However, from this we must subtract the number of ways there are to choose those four points such that three of them are colinear. There are 8 ways to choose three points such that all of them are colinear, and for all of them, there are 6 ways to choose the fourth point, so the number of ways to choose four points such that they can form a quadrilateral that actually has four distinct sides is $\binom{9}{4} - 6 \times 8 = 78$.

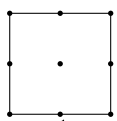
We are not done yet. Some of those sets of four points will give us exactly one quadrilateral (such as $(0,0),(0,1),(1,1),(1,0)$), while others (such as $(0,0),(1,2),(1,1),(2,1)$) give us three. A set of four points with no three of the points colinear will give us three quadrilaterals if one of the points is inside the triangle formed by the other three; otherwise, the four points give us one quadrilateral. There are exactly eight sets of points such that one is inside the triangle formed by the other three:

$$(1,1),(0,2),(1,0),(2,1)$$
$$(1,1),(2,2),(1,0),(0,1)$$
$$(1,1),(2,0),(0,1),(1,2)$$
$$(1,1),(0,0),(2,1),(1,2)$$
$$(1,1),(1,2),(0,0),(2,0)$$
$$(1,1),(2,1),(0,0),(0,2)$$
$$(1,1),(1,0),(0,2),(2,2)$$
$$(1,1),(0,1),(2,0),(2,2)$$

The other 70 possible sets of points all give us one quadrilateral. Therefore, the answer to the problem is $1(70) + 3(8) = 94$.

Solution 2 by: Keone Hon (10/HI)
There are **94** such quadrilaterals. We will separate the quadrilaterals into a number of cases as a primary means of enumerating them:

Rectangles: There are 10 rectangles in all: one 2×2, one $\sqrt{2} \times \sqrt{2}$, four 1×1s, and four 1×2s. An example of each is shown below: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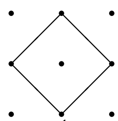
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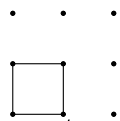
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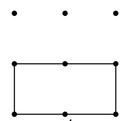
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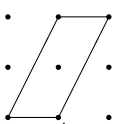
Parallelograms (other than rectangles): there are two types of parallelograms with non-perpendicular sides: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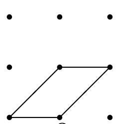
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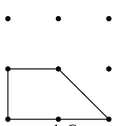
There are four parallelograms of the first shape, since the pair of currently horizontal sides may be any of the four pairs of opposite sides on an edge of the figure. There are eight parallelograms of the second shape, since for each of the four 1×2 rectangles, there are two such parallelograms inside. In all, there are then 12 non-rectangle parallelograms.

Trapezoids: There are three types of trapezoids: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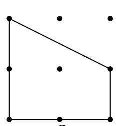
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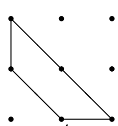
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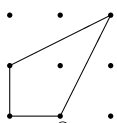
There are 16 trapezoids of the first type, since each of the four 1×2 rectangles contains four such trapezoids (each one is formed by cutting off a corner, and there are four corners). There are 8 trapezoids of the second type, since any of the four edges of the array can be the side that is currently on the bottom, and for each of those four choices, the longer of the two bases can be chosen from among two bases. There are four trapezoids of the third type, since the two non-parallel edges can (when extended) meet at any of the four corners of the array. In all, there are 28 trapezoids.

Kites: There is only one type of kite. There are four of this type, since the right-angle vertex can be chosen from any of the four corners of the array.



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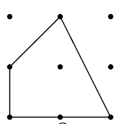
Other convex figures: There are two other types of convex figures with no parallel sides. For the first shape, any of the four edges of the array can be chosen as the side with length 2, and from there, any of the two adjacent edges can be chosen as the side with length 1. Thus there are 8 of this shape. For the second shape, any of the 8 segments on the edge of the array with length 1 can be chosen as the side with length 1; from there, the rest of the shape is determined. Thus there are 8 of this shape. In all, then, there are 16 such figures.



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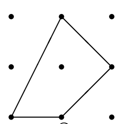

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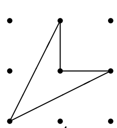
Concave figures: There are four types of convex figures. There are four of the first shape, since any of the four corners of the array can be chosen as the vertex currently in the bottom left corner. There are eight of the second shape, since in addition to the same four choices for the corner vertex, there are also two choices for which side the obtuse angle will open towards. There are eight of the third shape, since there are four choices for the side of length 2 (it can lie on any of the four edges of the array) and two choices for which side the obtuse angle will open towards. There are four of the last shape, since the obtuse angle can open towards any of the four edges of the array. In all, there are 24 concave figures.



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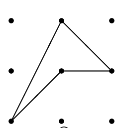

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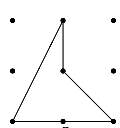

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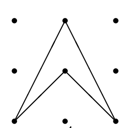

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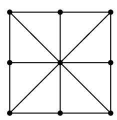
These are all the shapes. In all, there are $10 + 12 + 28 + 4 + 16 + 24 =$ **94 quadrilaterals**.

To verify that all the shapes have been counted exactly once, consider the following. There are $\binom{9}{4} = 126$ ways to choose 4 points out of the 9. Of all combinations of four points, the only ones that do not form quadrilaterals are where three points are collinear. Three points can be collinear if they are on any of the eight lines passing through three points, as shown below. The fourth point can be chosen from any of the other 6 points not on that line. Thus, there are 48 combinations of four points that cannot form quadrilaterals. So there are $126 - 48 = 78$ combinations of four points that do form quadrilaterals.



Given any collection of four noncollinear points, exactly one convex figure can be drawn through them. Thus, 70 combinations of four points correspond to 70 convex quadrilaterals. However, this is not true for concave figures; more than one concave figure can be drawn through four points. Each of the remaining eight combinations can be formed into three distinct quadrilaterals, as demonstrated below. Thus there are 24 concave quadrilaterals. In all, there are $70 + 24 =$ **94 quadrilaterals**, which agrees with our previous count.