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Solutions to Problem 4/1/19
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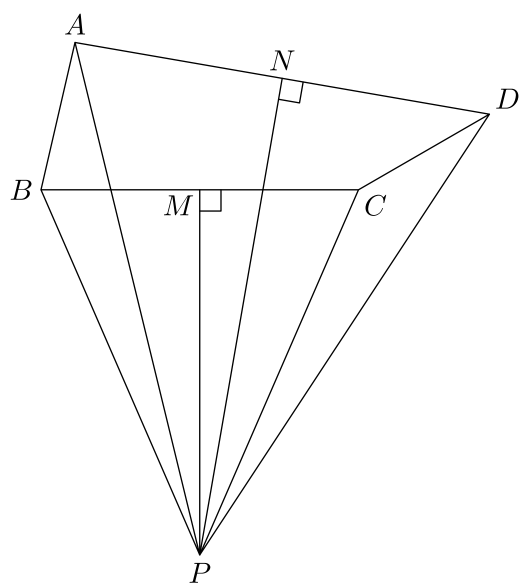
4/1/19. In convex quadrilateral $ABCD$, $AB = CD$, $\angle ABC = 77°$, and $\angle BCD = 150°$. Let P be the intersection of the perpendicular bisectors of $\overline{BC}$ and $\overline{AD}$. Find $\angle BPC$.

Credit This problem was proposed by Naoki Sato.

Comments Since P lies on the perpendicular bisector of BC, $PB = PC$. This and similar observations lead to the construction of congruent triangles which determine $\angle BPC$. In addition, the solution below rigorously estalishes the location of point P. *Solutions edited by Naoki Sato.*

Solution 1 by: Carl Lian (9/MA)



Note that there are three distinct cases for the position of P: Either outside quadrilateral $ABCD$ on the side of BC, that is, $PM < PN$; outside quadrilateral $ABCD$ on the side of AD, that is, $PN < PM$; or inside quadrilateral $ABCD$. We first deal with the first case, and then prove that the second and third cases are impossible.

Let M be the midpoint of BC and N the midpoint of AD. We have $BM = MC$ and $AN = ND$, and $\angle BMP = \angle CMP = \angle ANP = \angle DNP = 90°$, so $\triangle BMP \cong \triangle CMP$ and $\triangle ANP \cong \triangle DNP$. From these congruences, $BP = CP$ and $AP = DP$, and we are given that $AB = CD$. Therefore, $\triangle ABP \cong \triangle DCP$, and $\angle ABP = \angle DCP$.

Let $\theta = \angle CBP$. Then $\angle DCP = \angle ABP = 77° + \theta$, and $\angle BCP = \theta$. Now, by the angles around C, we have $\angle DCB + \angle BCP + \angle PCD = 150° + \theta + 77° + \theta = 360°$, so $2\theta = 133°$. Hence, $\angle BPC = 2\angle BPM = 2(90° - \theta) = 180° - 2\theta = 47°$.

For the second case, assume by way of contradiction that P lies outside quadrilateral $ABCD$, on the side of AD. Again, $\triangle ABP \cong \triangle DCP$. We have $\angle PBA = \angle PCD$, and also $\angle PBM = \angle PCM$ from $\triangle PBM \cong \triangle PCM$. Adding these gives $\angle PBA + \angle PMB = \angle PCD + \angle PCM$, and thus $\angle ABC = \angle BCD$, but this is a contradiction because $\angle ABC = 77^\circ$, and $\angle BCD = 150^\circ$, so P cannot lie outside quadrilateral $ABCD$ on the side of AD.

For the third case, assume by way of contradiction that P lies inside quadrilateral $ABCD$. Again, $\triangle ABP \cong \triangle DCP$. We have $\angle PBA = \angle PCD$, and also $\angle PBM = \angle PCM$ from $\triangle PBM \cong \triangle PCM$. Adding these gives $\angle PBA + \angle PMB = \angle PCD + \angle PCM$, and thus $\angle ABC = \angle BCD$, but this is a contradiction because $\angle ABC = 77^\circ$ and $\angle DBC = 150^\circ$, so P cannot lie inside quadrilateral $ABCD$.

Therefore, the first case is the only possible case, and our assertion that $\angle BPC = 47^\circ$ still holds.