4/1/18. Every point in the plane is colored either red, green, or blue. Prove that there exists a rectangle in the plane such that all four of its vertices are the same color.

Credit This problem was proposed by Dave Patrick, and comes from the Carnegie Mellon Mathematical Studies Problem Seminar.

Comments Once you have the idea of using a sufficiently large grid of points, the problem quickly reduces to an application of the Pigeonhole principle. Note to USAMTS students: It is more important to get the stated problem correct, before moving onto proving a generalization. *Solutions edited by Naoki Sato.*

Solution 1 by: Adam Hesterberg (12/WA)

Consider a 4×82 rectangle of points in the plane, such as $\{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x \leq 3,\ 0 \leq y \leq 81\}$. For each column, there are 4 points and 3 possible colors per point, for a total of $3^4 = 81$ possible colorings. With 82 columns, by the Pigeonhole Principle, there are two columns with the same coloring. Also, there are 4 points per column and 3 possible colors, so by the Pigeonhole Principle, some color appears twice. From each of the two columns, take some corresponding two points of a color that appears twice. These form a rectangle all of whose vertices are the same color.

Solution 2 by: Sam Elder (11/CO)

Consider a 4×19 grid of points in this plane. For each row of 4 points, by the Pigeonhole Principle, two must be the same color, for instance green. Denote such a row "green" (a row can be two colors simultaneously) and consider the colors of all 19 rows. Again by the Pigeonhole Principle, 7 must be the same color. Without loss of generality, assume this color is green.

Now consider the placement of the two green points out of four in each row. There are $\binom{4}{2} = 6$ ways to place two green points in four spots, so again by the Pigeonhole Principle, two of the seven rows must have the same placement. By choosing the four green points in those two rows, we form a monochromatic rectangle, as desired.