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Solutions to Problem 3/4/17
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3/4/17. We play a game. The pot starts at \$0. On every turn, you flip a fair coin. If you flip heads, I add \$100 to the pot. If you flip tails, I take all of the money out of the pot, and you are assessed a "strike." You can stop the game before any flip and collect the contents of the pot, but if you get 3 strikes, the game is over and you win nothing. Find, with proof, the expected value of your winnings if you follow an optimal strategy.

Credit This problem was proposed by Dave Patrick.

Comments This problem is similar to problems 2/1/17 and 2/3/17, in that both involved expected value, and both used the technique of reducing the problem to a simpler problem. This particular problem is best solved by considering in sequence what happens when you have one strike left, then two strikes, then the full three strikes. *Solutions edited by Naoki Sato.*

Solution 1 by: Wei Hao (11/VA)

Consider first the case when I already have two strikes. In order to find the optimal time to stop, let us assume that there are x dollars in the pot. I will toss again only when I will, on average, make more than x dollars, or

$$\frac{1}{2} \cdot (x + 100) + \frac{1}{2} \cdot 0 > x,$$

which implies

$$x < 100. \tag{1}$$

So if $x < 100$, it is advantageous to risk the last strike by tossing again. But the only possible value for $x < 100$ is $x = 0$. Therefore, with two strikes, I will toss once and stop the game no matter what the outcome is. The expected return for this toss is then \$50.

Consider next when I have only one strike. I can use the same logic to decide when to stop the game, except that if I get a tail, I have one more strike to give. As a result, I should use the \$50 expected value from the above discussion as the expected winnings if I get a tail. So, assuming again that there are x dollars in the pot before a toss, it is worthwhile to risk another strike only when

$$\frac{1}{2} \cdot (x + 100) + \frac{1}{2} \cdot 50 > x,$$

which gives

$$x < 150. \tag{2}$$

To find the expected winnings in this case, I will follow the following strategy: Since immediately after a strike, there is \$0 dollar in the pot, so I will flip the coin again. If I get a head, there will be \$100 dollars in the pot, and I still have only one strike. But \$100 is less than \$150, therefore, I can flip again. On the second flip, I will either get a head and stop the game because there will be \$200 in the pot, or I will get a tail and face the two strike problem discussed before. If I get a tail on the first flip, I will also face the same two strike problem. So the expected winnings with one strike is

$$\frac{1}{2}\left(\frac{1}{2}\cdot 200+\frac{1}{2}\cdot 50\right)+\frac{1}{2}\cdot 50=87.5. \tag{3}$$

Finally, I can use the same reasoning to decide when to stop the game when I have no strikes. The only difference is that in the event of getting a tail, I must use the result of equation (3) as the expected winnings since I will have one strike then. Assuming that there are x dollars in the pot to begin with, it is advantageous to try another flip when

$$\frac{1}{2}\cdot(x+100)+\frac{1}{2}\cdot 87.5>x,$$

or

$$x<187.5. \tag{4}$$

The expected winnings for following this strategy can be calculated in the same way as the one strike case. I start by flipping the coin. If I get a head, there will be \$100 in the pot. But that is less than the \$187.5 of equation (4). So I will flip again. If I get a head again, there will be \$200 in the pot and I will stop the game. But if I get a tail on the second flip, the problem is reduced to the one strike problem with an expected winnings of \$87.5. The same thing happens if I get a tail on the first flip. Therefore, the expected winnings for this game is

$$\frac{1}{2}\left(\frac{1}{2}\cdot 200+\frac{1}{2}\cdot 87.5\right)+\frac{1}{2}\cdot 87.5=115.625. \tag{5}$$

So the expected winnings for my strategy is \$115.625.