3/4/16. Find, with proof, a polynomial $f(x, y, z)$ in three variables, with integer coefficients, such that for all integers a, b, c, the sign of $f(a, b, c)$ (that is, positive, negative, or zero) is the same as the sign of $a + b\sqrt[3]{2} + c\sqrt[3]{4}$.

Credit This problem was devised by Dr. Erin Schram of the National Security Agency.

Comments Most students used algebraic manipulation to arrive at a solution; Tony Liu and Laura Starkston give example solutions.

Solutions edited by Richard Rusczyk.

Solution 1 by: Tony Liu (10/IL)

We claim that the polynomial $f(a, b, c) = a^3 + 2b^3 + 4c^3 - 6abc$ has the desired properties. Our proof begins with the following lemma:

Lemma: *The expressions $s = p^3 + q^3 + r^3 - 3pqr$ and $t = p + q + r$ have the same sign for real numbers p, q, r that are not all equal.*

Proof: We note the following identity:

$$\begin{aligned} p^3 + q^3 + r^3 - 3pqr &= (p + q + r)(p^2 + q^2 + r^2 - pq - qr - rp) \\ &= \frac{1}{2}(p + q + r)((p - q)^2 + (q - r)^2 + (r - p)^2), \end{aligned}$$

or equivalently,

$$s = \frac{t}{2}((p - q)^2 + (q - r)^2 + (r - p)^2).$$

Note that $(p-q)^2 + (q-r)^2 + (r-p)^2 \geq 0$, with equality if and only if $p = q = r$. By hypothesis, this cannot hold, so $(p-q)^2+(q-r)^2+(r-p)^2 > 0$. Thus, $t = 0$ if and only if $s = 0$. Moreover, when $s, t \neq 0$, we may divide by t to get $\frac{s}{t} = \frac{1}{2}((p - q)^2 + (q - r)^2 + (r - p)^2) > 0$, and the result follows. ■

Now, we set $p = a, q = b\sqrt[3]{2}$, and $r = c\sqrt[3]{4}$, so by our lemma,

$$p^3 + q^3 + r^3 - 3pqr = a^3 + 2b^3 + 4c^3 - 6abc$$

has the same sign as $p + q + r = a + b\sqrt[3]{2} + c\sqrt[3]{4}$, provided that $p = q = r$ does not hold. If $p = q = r$ does hold, then $a = b\sqrt[3]{2} = c\sqrt[3]{4}$, which implies $a = b = c = 0$ because a, b, c are integers. Thus, $f(a,b,c) = a^3 + 2b^3 + 4c^3 - 6abc = a + b + c = 0$, and this case is covered as well. This concludes our proof.

Solution 2 by: Laura Starkston (11/AZ)

If the signs must be the same, the zeros must be the same so...

$$\begin{aligned} a + b\sqrt[3]{2} + c\sqrt[3]{4} &= 0 \\ a\sqrt[3]{2} + b\sqrt[3]{4} + 2c &= 0 \\ -a\sqrt[3]{2} - b\sqrt[3]{4} &= 2c \end{aligned}$$

Keep this in mind. Rewrite the original equation:

$$\begin{aligned} a + b\sqrt[3]{2} + c\sqrt[3]{4} &= 0 \\ \left(a + b\sqrt[3]{2}\right)^3 &= \left(-c\sqrt[3]{4}\right)^3 \\ a^3 + 3\sqrt[3]{2}a^2b + 3\sqrt[3]{4}ab^2 + 2b^3 &= -4c^3 \\ a^3 + 2b^3 + 4c^3 &= -3\sqrt[3]{2}a^2b - 3\sqrt[3]{4}ab^2 \\ a^3 + 2b^3 + 4c^3 &= (-a\sqrt[3]{2} - b\sqrt[3]{4})(3ab) \end{aligned}$$

Combine the equations:

$$\begin{aligned} a^3 + 2b^3 + 4c^3 &= 6abc \\ a^3 + 2b^3 + 4c^3 - 6abc &= 0 \end{aligned}$$

Since the only operations performed were multiplication by a constant (which does not change the sign or the zeros, only the magnitude of the values) and cubing (which does not change the zeros; it makes each zero occur 3 times, but each zero is still the same; it preserves the sign because it is an odd number power), $f(a,b,c)$ where the function is defined as $f(x,y,z) = x^3 + 2y^3 + 4z^3 - 6xyz$ should have the same sign as $a + b\sqrt[3]{2} + c\sqrt[3]{2}$.