3/3/18. Three circles with radius 2 are drawn in a plane such that each circle is tangent to the other two. Let the centers of the circles be points A, B, and C. Point X is on the circle with center C such that $AX + XB = AC + CB$. Find the area of $\triangle AXB$.

Credit This problem was proposed by Richard Rusczyk.

Comments The condition $AX + XB = AC + CB$ leads to X lying on an ellipse, whose equation can be determined with analytic geometry. *Solutions edited by Naoki Sato.*

Solution 1 by: Tony Liu (12/IL)

Position the three circles in the coordinate plane so that $A = (-2, 0)$ and $B = (2, 0)$, and let O be the origin. We can easily calculate $CO = \sqrt{CA^2 - OA^2} = 2\sqrt{3}$, so let $C = (0, 2\sqrt{3})$. Now, note that the locus of all points X such that $AX + XB = AC + CB$ is an ellipse $\mathcal{E}$.

More specifically, $\mathcal{E}$ is centered at O and has foci at A and B. If we let $A' = (-4, 0)$ and $B' = (4, 0)$, then $AA' + A'B = AB' + B'B = 8$, so A' and B' lie on the ellipse as well. The ellipse passes through C as well, so the equation of $\mathcal{E}$ is given by

$$\frac{x^2}{16} + \frac{y^2}{12} = 1 \quad \Rightarrow \quad x^2 + \frac{4y^2}{3} = 16.$$

To locate X, we want to find the point at which the ellipse $\mathcal{E}$ intersects the circle centered at C, which has the equation

$$x^2 + (y - 2\sqrt{3})^2 = 4.$$

Substituting, we obtain

$$4 - (y - 2\sqrt{3})^2 + \frac{4y^2}{3} = 16 \quad \Rightarrow \quad \frac{y^2}{3} + 4\sqrt{3}y = 24.$$

Solving this quadratic and taking the positive root yields $y = 6\sqrt{5} - 6\sqrt{3}$. This is equal to the altitude of triangle AXB, with respect to base AB, so the area of this triangle is simply $\frac{1}{2} \cdot 4 \cdot y = 12\sqrt{5} - 12\sqrt{3}$.