3/1/19. Find all positive integers $a \le b \le c$ such that

$$\arctan \frac{1}{a} + \arctan \frac{1}{b} + \arctan \frac{1}{c} = \frac{\pi}{4}.$$

Credit This problem was proposed by Naoki Sato.

Comments First, we can use the properties of the arctan function to establish bounds on a. Then we can transform the given equation into an algebraic equation, from which we can deduce the solutions. *Solutions edited by Naoki Sato.*

Solution 1 by: Damien Jiang (10/NC)

We first establish bounds on a. Since $\arctan x$ is increasing on $(0, 1]$,

$$\arctan \frac{1}{a} \ge \arctan \frac{1}{b} \ge \arctan \frac{1}{c}.$$

Hence,

$$\frac{\pi}{4} = \arctan \frac{1}{a} + \arctan \frac{1}{b} + \arctan \frac{1}{c} \le 3 \arctan \frac{1}{a},$$

so

$$\arctan \frac{1}{a} \ge \frac{\pi}{12} \quad \Rightarrow \quad \frac{1}{a} \ge \tan \frac{\pi}{12} = 2 - \sqrt{3} \quad \Rightarrow \quad a \le \frac{1}{2 - \sqrt{3}} = 2 + \sqrt{3} < 4.$$

Additionally,

$$\frac{\pi}{4} = \arctan \frac{1}{a} + \arctan \frac{1}{b} + \arctan \frac{1}{c} > \arctan \frac{1}{a},$$

so

$$\frac{1}{a} < \tan \frac{\pi}{4} = 1 \quad \Rightarrow \quad a > 1.$$

Therefore, the only possible values of a are $a = 2$ and $a = 3$.

From the original equation, we subtract $\arctan \frac{1}{c}$, and take the tangent of both sides to get

$$\frac{\frac{1}{a} + \frac{1}{b}}{1 - \frac{1}{ab}} = \frac{1 - \frac{1}{c}}{1 + \frac{1}{c}}.$$

Note that this equation is equivalent with the original because $\tan x$ is injective on $(0, 1]$. Multiplying, clearing denominators, and rearranging, we get

$$abc + 1 = ab + ac + bc + a + b + c.$$

If $a = 2$, then

$$\begin{aligned} 2bc + 1 &= 2(b+c) + bc + 2 + b + c \\ \Rightarrow \quad bc - 3(b+c) &= 1 \\ \Rightarrow \quad (b-3)(c-3) &= 10. \end{aligned}$$

Because $c > b$, we have $b = 4$, $c = 13$ or $b = 5$, $c = 8$.

If $a = 3$, then

$$\begin{aligned} 3bc + 1 &= 3(b+c) + bc + 3 + b + c \\ \Rightarrow \quad 2bc - 4(b+c) &= 2 \\ \Rightarrow \quad (b-2)(c-2) &= 5. \end{aligned}$$

Because $c > b$, we have $b = 3$, $c = 7$.

Therefore, the only solutions are $(a, b, c) = (2, 4, 13)$, $(2, 5, 8)$, and $(3, 3, 7)$.