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Solutions to Problem 3/1/17
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3/1/17. Let r be a nonzero real number. The values of z which satisfy the equation

$$r^4z^4 + (10r^6 - 2r^2)z^2 - 16r^5z + (9r^8 + 10r^4 + 1) = 0$$

are plotted on the complex plane (i.e. using the real part of each root as the x-coordinate and the imaginary part as the y-coordinate). Show that the area of the convex quadrilateral with these points as vertices is independent of r, and find this area.

Credit This problem was proposed by Dave Patrick of AoPS and Erin Schram of the NSA.

Comments Many students seem to have been intimidated by the complicated looking quartic equation and the setting of the complex plane. The first step is to find the factors of the quartic. This is really the bulk of the problem, and was accomplished with a variety of approaches, as the following solutions illustrate. The next step is to plot the roots in the complex plane, which are found to form a trapezoid. Some students merely plugged the quartic into software such as *Mathematica*, but you still need to show justification that the roots so produced are in fact correct. *Solutions edited by Naoki Sato.*

Solution 1 by: Daniel Jiang (11/IN)

The constant coefficient of the equation can be factored so that the equation becomes:

$$r^4z^4 + (10r^6 - 2r^2)z^2 - 16r^5z + (9r^4 + 1)(r^4 + 1) = 0.$$

Factoring would lead us to the roots of the equation as a function of r. From what we have so far, we can see that the factors of the equation may look like

$$[r^2z^2 + \cdots + (r^4 + 1)][r^2z^2 + \cdots + (9r^4 + 1)].$$

The given equation has powers of z^4, z^2, and z, so at this stage, we let the factors take the form

$$[r^2z^2 + az + (r^4 + 1)][r^2z^2 + bz + (9r^4 + 1)].$$

Expanding, we get:

$$r^4z^4 + (ar^2 + br^2)z^3 + (10r^6 + 2r^2 + ab)z^2 + (a + b + 9ar^4 + br^4)z + (9r^8 + 10r^4 + 1) = 0.$$

Comparing this to the given equation, we know that there is no z^3 term, so $a + b = 0$, and from the z^2 term, we see that $ab = -4r^2$. Using these, it is easy to see that $(a, b) = (-2r, 2r)$ or $(2r, -2r)$, but then testing in the z term, we see that a must be $-2r$, so $(a, b) = (-2r, 2r)$.

The factored form of the equation is then

$$(r^2z^2 - 2rz + r^4 + 1)(r^2z^2 + 2rz + 9r^4 + 1) = 0.$$

Now the quadratic formula can be applied to each of the quadratic factors, which gives us the four roots as: $\frac{1}{r} \pm ri$ and $-\frac{1}{r} \pm 3ri$.

Plotting the points $\left(\frac{1}{r}, r\right), \left(\frac{1}{r}, -r\right), \left(-\frac{1}{r}, 3r\right), \left(-\frac{1}{r}, -3r\right)$, we have a trapezoid with two bases of length $|6r|$ and $|2r|$ and a height of $\frac{2}{|r|}$. The area of a trapezoid is $\frac{b_1+b_2}{2} \cdot h$, so the area is $\frac{|8r|}{2} \cdot \frac{2}{|r|} = 8$. The r cancels out; thus, the area of this convex quadrilateral is 8 independent of r.

Solution 2 by: Joshua Horowitz (11/CT)

The given expression can be written and factored as the difference of two squares:

$$\begin{aligned}
r^4z^4 + (10r^6 - 2r^2)z^2 - 16r^5z + (9r^8 + 10r^4 + 1) &= 0 \\
\Rightarrow\ [r^2z^2 + (5r^4+1)]^2 - (2rz + 4r^4)^2 &= 0 \\
\Rightarrow\ [r^2z^2 + 2rz + (9r^4+1)][r^2z^2 - 2rz + (r^4+1)] &= 0.
\end{aligned}$$

So the roots of the original equation are the roots of $P(z) = r^2z^2 + 2rz + (9r^4+1)$ combined with the roots of $Q(z) = r^2z^2 - 2rz + (r^4+1)$. Each of these is a quadratic with real coefficients. The quadratic P has discriminant $-36r^6$ and the quadratic Q has discriminant $-4r^6$. Both of these are negative (since r is a nonzero real) so the roots of P and the roots of Q form conjugate pairs.

Using the quadratic formula we can compute a root $p = -\frac{1}{r} + 3ri$ of P and a root $q = \frac{1}{r} + ri$ of Q. These two roots and their conjugates will form an isosceles trapezoid with the real axis as an axis of symmetry. The area of this trapezoid (the desired answer to this problem) will be the height times the sum of half the bases:

$$|\operatorname{Re} p - \operatorname{Re} q|\,(|\operatorname{Im} p| + |\operatorname{Im} q|) = \left|\frac{2}{r}\right|(|3r| + |r|) = \left|\frac{2}{r}\right||4r| = 8,$$

where $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary parts of the complex number z, respectively.

Solution 3 by: Linda Liu (11/GA)

We have that

$$\begin{aligned}
r^4z^4 + (10r^6 - 2r^2)z^2 - 16r^5z + (9r^8 + 10r^4 + 1) &= 0 \\
\Rightarrow\ r^4z^4 + (6r^6 - 2r^2)z^2 + 9r^8 - 6r^4 + 1 + 4r^6z^2 - 16r^5z + 16r^4 &= 0 \\
\Rightarrow\ r^4z^4 + 2(3r^4-1)r^2z^2 + (3r^4-1)^2 + (2r^3z - 4r^2)^2 &= 0 \\
\Rightarrow\ (r^2z^2 + 3r^4 - 1)^2 + (2r^3z - 4r^2)^2 &= 0,
\end{aligned}$$

so

$$(r^2z^2 + 3r^4 - 1)^2 = -(2r^3z - 4r^2)^2.$$

Taking the square root of both sides gives the two equations

$$r^2z^2 + 3r^4 - 1 = (2r^3z - 4r^2)i \;\Rightarrow\; r^2z^2 - 2r^3zi + 3r^4 + 4r^2i - 1 = 0,$$

and

$$r^2z^2 + 3r^4 - 1 = -(2r^3z - 4r^2)i \;\Rightarrow\; r^2z^2 + 2r^3zi + 3r^4 - 4r^2i - 1 = 0.$$

Applying the quadratic formula to the first quadratic equation produces the roots

$$-\frac{1}{r} + 3ri \quad \text{and} \quad \frac{1}{r} - ri.$$

Applying the quadratic formula to the second quadratic equation produces the roots

$$-\frac{1}{r} - 3ri \quad \text{and} \quad \frac{1}{r} + ri.$$

These four complex numbers then form a trapezoid with height $2/|r|$ and bases $|2r|$ and $|6r|$, so the area of the trapezoid is

$$\frac{|2r| + |6r|}{2} \times \frac{2}{r} = 8.$$