
2/4/19. Determine, with proof, the greatest integer n such that

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{11} \right\rfloor + \left\lfloor \frac{n}{13} \right\rfloor < n,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Credit This problem was proposed by Andy Niedermaier.

Comments In an intuitive (but not rigorous) sense, n should leave the maximum remainder when divided by 2, 3, 11, and 13, i.e. n should leave a remainder of 12 when divided by 13, and so on. The following proof rigorously establishes the answer by finding a bound in terms of these remainders. *Solutions edited by Naoki Sato.*

Solution by: Wenyu Cao (9/NJ)

We claim that the greatest integer that satisfies the given inequality is 1715. First, we check that $n = 1715$ satisfies the given inequality:

$$\left\lfloor \frac{1715}{2} \right\rfloor + \left\lfloor \frac{1715}{3} \right\rfloor + \left\lfloor \frac{1715}{11} \right\rfloor + \left\lfloor \frac{1715}{13} \right\rfloor = 857 + 571 + 155 + 131 = 1714 < 1715.$$

Next, we claim that

$$\left\lfloor \frac{x}{k} \right\rfloor \geq \frac{x - k + 1}{k}$$

for all positive integers x and k. By the Division Algorithm, there exist integers q and r such that $x = qk + r$ and $0 \leq r \leq k - 1$. Then

$$\left\lfloor \frac{x}{k} \right\rfloor = \left\lfloor \frac{qk + r}{k} \right\rfloor = \left\lfloor q + \frac{r}{k} \right\rfloor = q,$$

since $0 \leq r/k < 1$, and

$$\frac{x - k + 1}{k} = \frac{qk + r - k + 1}{k} = q + \frac{r - (k - 1)}{k} \leq q = \left\lfloor \frac{x}{k} \right\rfloor,$$

as desired.

Now, let n be a positive integer that satisfies the given inequality:

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{11} \right\rfloor + \left\lfloor \frac{n}{13} \right\rfloor < n.$$

Since both sides of the inequality are integers,

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{11} \right\rfloor + \left\lfloor \frac{n}{13} \right\rfloor \leq n - 1.$$

Therefore, from the result above,

$$\begin{aligned} n-1 &\geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{11} \right\rfloor + \left\lfloor \frac{n}{13} \right\rfloor \\ &\geq \frac{n-1}{2} + \frac{n-2}{3} + \frac{n-10}{11} + \frac{n-12}{13} \\ &= \frac{859}{858}n - \frac{2573}{858}. \end{aligned}$$

Then

$$\frac{n}{858} \leq \frac{1715}{858},$$

so $n \leq 1715$. Since we have shown that $n = 1715$ works, we are done.