USA Mathematical Talent Search

Solutions to Problem 2/3/19

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2/3/19. Gene starts with the 3×3 grid of 0's shown at left below. He then repeatedly chooses a 2×2 square within the grid and increases all four numbers in the chosen 2×2 square by 1. One possibility for Gene's first three steps is shown below.

0	0	0
0	0	0
0	0	0

→

0	0	0
1	1	0
1	1	0

→

0	1	1
1	2	1
1	1	0

→

0	1	1
2	3	1
2	2	0

How many different grids can be produced with this method such that each box contains an integer from 1 to 12, inclusive? (The numbers in the boxes need not be distinct.)

Comments The number of different grids can be counted by identifying each grid with a quintuple of positive integers. Then the number of such quintuples can be found using a standard partition argument. *Solutions edited by Naoki Sato.*

Solution by: Matt Superdock (11/PA)

Any 2×2 square inside the 3×3 grid includes the center box, so the integer in the center box is equal to the number of steps Gene makes. Therefore, Gene can make at most 12 steps, or else the middle box will contain an integer greater than 12. Additionally, each corner box of the 3×3 grid is included in only one of the four 2×2 squares. Therefore, Gene must choose each 2×2 square at least once, or else one of the corner boxes will contain a 0. The final grid depends only on how many times Gene chooses each 2×2 square, not which order he chooses them.

Let a, b, c, and d be the numbers of times Gene chooses each 2×2 square. Then a, b, c, and d are positive integers, and $a + b + c + d \leq 12$. Let $e = 13 - a - b - c - d$, so that e is also a positive integer, and $a + b + c + d + e = 13$. To find the number of solutions to this equation, we consider partitioning 13 objects into 5 non-empty groups. We arrange the 13 objects in a row, and we partition them by placing dividers in 4 of the 12 spaces between adjacent objects. There are $\binom{12}{4} = 495$ ways to do this, so there are 495 grids that can be produced.