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Solutions to Problem 2/2/17
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2/2/17. Write the number

$$\frac{1}{\sqrt{2}-\sqrt[3]{2}}$$

as the sum of terms of the form 2^q, where q is rational. (For example, $2^1 + 2^{-1/3} + 2^{8/5}$ is a sum of this form.) Prove that your sum equals $1/(\sqrt{2}-\sqrt[3]{2})$.

Credit This problem was proposed by Sidney Kravitz.

Comments This problem is effectively an exercise in rationalizing the denominator, but the twist is the presence of both the square root and the cube root. A knowledge of basic algebraic identities can take care of both. James Sundstrom arrives at the answer using two steps, Justin Hsu shows how to the same calculation in one step, and Jeffrey Manning provides a clever solution using geometric series. *Solutions edited by Naoki Sato.*

Solution 1 by: James Sundstrom (11/NJ)

The identities $(a-b)(a+b) = a^2 - b^2$ and $(a-b)(a^2+ab+b^2) = a^3 - b^3$ suggest the following approach for rationalizing the denominator. For example, setting $a = \sqrt{2}$ and $b = \sqrt[3]{2}$ gives

$$(\sqrt{2}-\sqrt[3]{2})(\sqrt{2}+\sqrt[3]{2}) = 2 - \sqrt[3]{4},$$

so

$$\frac{1}{\sqrt{2}-\sqrt[3]{2}} = \frac{\sqrt{2}+\sqrt[3]{2}}{(\sqrt{2}-\sqrt[3]{2})(\sqrt{2}+\sqrt[3]{2})} = \frac{\sqrt{2}+\sqrt[3]{2}}{2-\sqrt[3]{4}}.$$

Then setting $a = 2$ and $b = \sqrt[3]{4}$ gives

$$(2-\sqrt[3]{4})(4+2\sqrt[3]{4}+2\sqrt[3]{2}) = 8 - 4 = 4,$$

so

$$\begin{aligned}
\frac{\sqrt{2}+\sqrt[3]{2}}{2-\sqrt[3]{4}} &= \frac{(\sqrt{2}+\sqrt[3]{2})(4+2\sqrt[3]{4}+2\sqrt[3]{2})}{(2-\sqrt[3]{4})(4+2\sqrt[3]{4}+2\sqrt[3]{2})} \\
&= \frac{4\sqrt{2}+4\sqrt[3]{2}+2\sqrt[3]{4}\sqrt{2}+4+2\sqrt[3]{2}\sqrt{2}+2\sqrt[3]{4}}{4} \\
&= \sqrt{2}+\sqrt[3]{2}+\frac{\sqrt[3]{4}\sqrt{2}}{2}+1+\frac{\sqrt[3]{2}\sqrt{2}}{2}+\frac{\sqrt[3]{4}}{2} \\
&= 2^{\frac{1}{2}}+2^{\frac{1}{3}}+2^{\frac{2}{3}}\left(2^{\frac{1}{2}}\right)\left(2^{-1}\right)+2^0+2^{\frac{1}{3}}\left(2^{\frac{1}{2}}\right)\left(2^{-1}\right)+2^{\frac{2}{3}}\left(2^{-1}\right) \\
&= 2^{\frac{1}{2}}+2^{\frac{1}{3}}+2^{\frac{1}{6}}+2^0+2^{-\frac{1}{6}}+2^{-\frac{1}{3}}.
\end{aligned}$$

Solution 2 by: Justin Hsu (11/CA)

We have the identity

$$(a-b)(a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5)=a^6-b^6.$$

Now, we let $a=\sqrt{2}$ and $b=\sqrt[3]{2}$, and multiply the numerator and denominator by $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$. This gives us

$$\begin{aligned}
&\frac{1}{\sqrt{2}-\sqrt[3]{2}}\cdot\frac{4\sqrt{2}+4\sqrt[3]{2}+2\sqrt[3]{4}\sqrt{2}+4+2\sqrt[3]{2}\sqrt{2}+2\sqrt[3]{4}}{4\sqrt{2}+4\sqrt[3]{2}+2\sqrt[3]{4}\sqrt{2}+4+2\sqrt[3]{2}\sqrt{2}+2\sqrt[3]{4}}\\
&=\frac{4\sqrt{2}+4\sqrt[3]{2}+2\sqrt[3]{4}\sqrt{2}+4+2\sqrt[3]{2}\sqrt{2}+2\sqrt[3]{4}}{4}\\
&=2^{\frac{1}{2}}+2^{\frac{1}{3}}+2^{\frac{1}{6}}+2^0+2^{-\frac{1}{6}}+2^{-\frac{1}{3}},
\end{aligned}$$

giving us the required sum of rational powers of 2.

Solution 3 by: Jeffrey Manning (10/CA)

Notice that if $x\neq 0$, then the sum $x^3+x^2+x+1+x^{-1}+x^{-2}$ is a geometric series with common ratio x^{-1} and initial term x^3. If $x\neq 1$, then we can use the formula for a geometric series to get

$$x^3+x^2+x+1+x^{-1}+x^{-2}=\frac{x^3(1-x^{-6})}{1-x^{-1}}=\frac{x^3-x^{-3}}{1-x^{-1}}=\frac{(x^3-x^{-3})x^3}{(1-x^{-1})x^3}=\frac{x^6-1}{x^3-x^2}.$$

Applying this formula with $x=2^{1/6}$ gives

$$\begin{aligned}
\frac{1}{\sqrt{2}-\sqrt[3]{2}}&=\frac{2^1-1}{2^{1/2}-2^{1/3}}\\
&=\frac{(2^{1/6})^6-1}{(2^{1/6})^3-(2^{1/6})^2}\\
&=(2^{1/6})^3+(2^{1/6})^2+2^{1/6}+1+(2^{1/6})^{-1}+(2^{1/6})^{-2}\\
&=2^{1/2}+2^{1/3}+2^{1/6}+2^0+2^{-1/6}+2^{-1/3}.
\end{aligned}$$