2/2/16. Call a number $a - b\sqrt{2}$ with a and b both positive integers *tiny* if it is closer to zero than any number $c - d\sqrt{2}$ such that c and d are positive integers with $c < a$ and $d < b$. Three numbers which are tiny are $1 - \sqrt{2}$, $3 - 2\sqrt{2}$, and $7 - 5\sqrt{2}$. Without using a calculator or computer, prove whether or not each of the following is tiny:

$$(a)\ 58 - 41\sqrt{2}\,, \qquad\qquad (b)\ 99 - 70\sqrt{2}\,.$$

Credit We are indebted to Dr. David Grabiner of the NSA for this problem. David is a former multiple winner of the USAMO, whose continued support of the USAMTS is most appreciated.

Comments Solution 1 shows the most straightforward solution. Solution 2 uses the shape of the graph of $y = \sqrt{x}$. Solution 3 uses the continued fraction representation of $\sqrt{2}$. Other solutions are possible, including listing (by hand!) all of the smallest numbers of the form $|a - b\sqrt{2}|$ for each positive integer a up through 100.

Solution 1 by: Tony Liu (10/IL)

(a) We claim that $58 - 41\sqrt{2}$ is not tiny. Indeed, from $1 < \sqrt{2}$, we have

$$\begin{aligned} |58 - 41\sqrt{2}| &> \frac{|58 - 41\sqrt{2}|}{\sqrt{2}} \\ &= |29\sqrt{2} - 41| \\ &= |41 - 29\sqrt{2}| \end{aligned}$$

Thus $41 - 29\sqrt{2}$ is closer to zero than $58 - 41\sqrt{2}$. Since $41 < 58$, and $29 < 41$, we conclude that $58 - 41\sqrt{2}$ is not tiny.

(b) We claim that $99 - 70\sqrt{2}$ is tiny. Assume, for the sake of contradiction, that there exists a number $c - d\sqrt{2}$ closer to zero, with $c < 99$, and $d < 70$. Since $99^2 - 2 \cdot 70^2 = 9801 - 9800 = 1$, we have

$$\begin{aligned} 1 &= |(99 - 70\sqrt{2})(99 + 70\sqrt{2})| \\ &> |(c - d\sqrt{2})(99 + 70\sqrt{2})| \\ &> |(c - d\sqrt{2})(c + d\sqrt{2})| \\ &= |c^2 - 2d^2| \end{aligned}$$

Because c and d are positive integers, this implies that $c^2 - 2d^2 = 0$, or $c = d\sqrt{2}$, which is impossible. It follows that $99 - 70\sqrt{2}$ is indeed tiny.

Solution 2 by: Johnny Hu (10/AL)

The three examples for *tiny* numbers are all in the form, $\sqrt{x+1} - \sqrt{x}$ or $\sqrt{x} - \sqrt{x+1}$, where x is an integer. Since the graph for $\sqrt{x}$ is half a parabola that opens to the positive side that rises more and more slowly as x increases, the difference between $\sqrt{x+1}$ and $\sqrt{x}$ becomes smaller and smaller as x increases. Since $x+1$ and x are consecutive integers and the difference between $\sqrt{x+1}$ and $\sqrt{x}$ becomes smaller as x increases, numbers in the form of $\sqrt{x+1} - \sqrt{x}$ and $\sqrt{x} - \sqrt{x+1}$ must be *tiny* because all values smaller than x will not produce a number closer to zero.

Since $58 - 41\sqrt{2}$ can be written as $\sqrt{3364} - \sqrt{3362}$, it is in the form of $\sqrt{x+2} - \sqrt{x}$. The graph of $\sqrt{x+2} - \sqrt{x}$ is above the graph of $\sqrt{x+1} - \sqrt{x}$ so $58 - 41\sqrt{2}$ is not a tiny number as there exists a number in the form of $c - d\sqrt{2}$, where $c < 58$ and $d < 41$, which is closer to zero.
To verify this, we must find a number in the form of $\sqrt{y+1} - \sqrt{y}$ or $\sqrt{y} - \sqrt{y+1}$, since these will most likely to be smaller than $\sqrt{x+2} - \sqrt{x}$ (This will be proven later in the page).
$a - b\sqrt{2} = 58 - 41\sqrt{2}$

$$\begin{aligned} 58 - 41\sqrt{2} &= a - b\sqrt{2} \\ &= \sqrt{a^2} - \sqrt{2b^2} \end{aligned}$$

Also:

$$\begin{aligned} a^2 &= 2b^2 + 2 \\ &= 2(b^2 + 1) \end{aligned}$$

To make this equation into the form of $\sqrt{y} - \sqrt{y+1}$:
Let:

$$(d)(\sqrt{2}) = a$$

Then:

$$2d^2 = 2(b^2 + 1)$$

$$d^2 = b^2 + 1$$

$$b^2 = y$$

$$d^2 = y + 1$$

Therefore, $\sqrt{y}-\sqrt{y+1}=\sqrt{b^2}-\sqrt{d^2}$
Substituting our original value, we have $\sqrt{1681}-\sqrt{1682}=41-29\sqrt{2}$.
To verify that $41-29\sqrt{2}$ is closer to zero than $58-41\sqrt{2}$:

$$|41-29\sqrt{2}|<|58-41\sqrt{2}|$$

$$|41-29\sqrt{2}|^2<|58-41\sqrt{2}|^2$$

$$3363-2378\sqrt{2}<6726-4756\sqrt{2}$$

Since $2(3363-2378\sqrt{2})=6726-4756\sqrt{2}$, $|41-29\sqrt{2}|<|58-41\sqrt{2}|$ and $58-41\sqrt{2}$ is not a *tiny* number.

Since $99-70\sqrt{2}$ can be written as $\sqrt{9801}-\sqrt{9800}$, it is in the form of $\sqrt{x+1}-\sqrt{x}$. Numbers in this form are always *tiny* numbers, so $99-70\sqrt{2}$ is a *tiny* number.

Solution 3 by: Zachary Abel (11/TX)

This problem follows from a (well known?) theorem concerning the approximation ability of continued fractions.

Theorem. *For a given irrational number α, the number $p-q\alpha$ is* tiny *if and only if p/q is a convergent of α.*

The proof is in two parts.

Lemma 1. *If p_n/q_n is a convergent for the irrational number α and $p/q\neq p_n/q_n$ is an arbitrary fraction with $0<q<q_{n+1}$, then*

$$|p_n-q_n\alpha|<|p-q\alpha|.$$

Proof. The key to this proof is to try to write

$$(p_n-q_n\alpha)x+(p_{n+1}-q_{n+1}\alpha)y=p-q\alpha$$

by solving the system

$$\begin{cases} q_nx+q_{n+1}y &= q \\ p_nx+p_{n+1}y &= p \end{cases} \tag{1}$$

for x and y. Using the fact that $p_{n+1}q_n-p_nq_{n+1}=(-1)^n$, we find from the above system that

$$x=(-1)^n\big(qp_{n+1}-pq_{n+1}\big)\quad\text{and}\quad y=(-1)^n\big(pq_n-qp_n\big)$$

This tells us a lot! First of all, both x and y are integers. Next, neither x nor y is 0. Indeed, if $x=0$ then $p/q=p_{n+1}/q_{n+1}$, which is impossible for $q<q_{n+1}$ since $\gcd(p_{n+1},q_{n+1})=1$, and if $y=0$ then $p/q=p_n/q_n$, which was assumed to be false.

We can obtain even more information from the sytem in (1): x and y must have opposite sign. If both were positive, then $q = q_n x + q_{n+1} y > q_{n+1}$, and if both were negative, then q would be negative.

Now we're ready for the final step. Since α lies between p_n/q_n and p_{n+1}/q_{n+1}, the numbers $p_n - q_n\alpha$ and $p_{n+1} - q_{n+1}\alpha$ have opposite signs. Since x and y also have opposite signs, the two numbers $(p_n - q_n\alpha)x$ and $(p_{n+1} - q_{n+1}\alpha)y$ have the same sign. Thus,

$$\begin{aligned}
(p_n - q_n\alpha)x + (p_{n+1} - q_{n+1}\alpha)y &= p - q\alpha \\
\big|(p_n - q_n\alpha)x + (p_{n+1} - q_{n+1}\alpha)y\big| &= \big|p - q\alpha\big| \\
\big|(p_n - q_n\alpha)x\big| + \big|(p_{n+1} - q_{n+1}\alpha)y\big| &= \big|p - q\alpha\big| \\
\big|p_n - q_n\alpha\big| \cdot \big|x\big| &< \big|p - q\alpha\big| \\
\big|p_n - q_n\alpha\big| &< \big|p - q\alpha\big|
\end{aligned}$$

□

Notice that this lemma shows that the number $p_n - q_n\alpha$ is *tiny*. This next lemma shows that there are no other tiny numbers.

Lemma 2. *If p/q is not a convergent of α, then $p - q\alpha$ is not tiny.*

Proof. Since p/q isn't a convergent to α, we can find two successive convergents p_n/q_n and p_{n+1}/q_{n+1} with $q_n < q < q_{n+1}$. Then by the first lemma, $|p_n - q_n\alpha| < |p - q\alpha|$, and so $p - q\alpha$ is not *tiny*. □

These two lemmas show that $p_n - q_n\alpha$ is tiny for each n and that there are no other tiny numbers. So the main theorem has been proven.

Because of this theorem with $\alpha = \sqrt{2}$, the tiny numbers can be found by calculating the convergents of $\sqrt{2}$. The continued fraction representation of $\sqrt{2}$ is

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

Using the recurrence relations

$$\begin{aligned}
p_n &= a_n p_{n-1} + p_{n-2} \\
q_n &= a_n q_{n-1} + q_{n-2}
\end{aligned}$$

and the fact that $a_n = 2$ for $n \geq 1$, we can easily calculate the convergents. We get

$$\frac{p_0}{q_0} = \frac{1}{1} \qquad \frac{p_1}{q_1} = \frac{3}{2} \qquad \frac{p_2}{q_2} = \frac{7}{5} \qquad \frac{p_3}{q_3} = \frac{17}{12}$$
$$\frac{p_4}{q_4} = \frac{41}{29} \qquad \frac{p_5}{q_5} = \frac{99}{70} \qquad \frac{p_6}{q_6} = \frac{239}{169} \qquad \frac{p_7}{q_7} = \frac{577}{408}$$

Since $99/70$ is one of the convergents, $99 - 70\sqrt{2}$ is a tiny number, whereas $58/41$ is not a convergent and so $58 - 41\sqrt{2}$ is not tiny.