USA Mathematical Talent Search

Solutions to Problem 2/1/18

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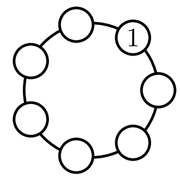
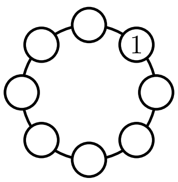
2/1/18.



(a) In how many different ways can the six empty circles in the diagram at right be filled in with the numbers 2 through 7 such that each number is used once, and each number is either greater than both its neighbors, or less than both its neighbors?



(b) In how many different ways can the seven empty circles in the diagram at right be filled in with the numbers 2 through 8 such that each number is used once, and each number is either greater than both its neighbors, or less than both its neighbors?

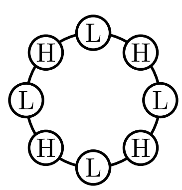
Credit This problem was proposed by Richard Rusczyk.

Comments This problem may be solved by dividing into cases, by considering which numbers must be greater than or less than their neighbors. *Solutions edited by Naoki Sato.*

Solution 1 by: Aaron Pribadi (11/AZ)

(a) Starting from a position in the loop, if the next value is greater, then the third value is less than the second, because the second value must be greater than both the first and third. If the next value is less, then the third value is greater than the second, because the second value must be less than both the first and third. This creates a higher, lower, higher, lower pattern. This pattern cannot exist with an odd number of positions; therefore, it is impossible to complete a seven-position diagram.

(b) Designate four 'high' positions as those greater than their neighbors, and four 'low' that are less than their neighbors, in an alternating pattern.



Neither 1 nor 2 may be in a 'high' position because it would require two numbers lower than it. Similarly, neither 7 nor 8 may be in a low position. So, there are $\binom{4}{2} = 6$ ways of dividing the numbers 1 to 8 into the two groups 'high' and 'low.'

Case 1. Low: 1, 2, 3, 4 High: 5, 6, 7, 8

Because all of the lows are less than all of the highs and all of the highs are greater than all of the lows, any low may be in any of the four low positions, and any high may be in any high position. Because the 1 is in four possible positions rather than one fixed position, there is a uniform overcounting by a factor of four. Therefore, for this case there are

$$\frac{4! \times 4!}{4} = 144$$

possible arrangements.

Case 2. Low: 1, 2, 3, 5 High: 4, 6, 7, 8

For a given arrangement of the lows (4! arrangements), the 4 cannot be next to the 5, so it has 2 possible positions. The other three highs may be in any of the three remaining positions and still produce a valid arrangement (3! arrangements). Therefore, there are

$$\frac{4! \times 2 \times 3!}{4} = 72$$

possible arrangements.

Case 3. Low: 1, 2, 4, 5 High: 3, 6, 7, 8

For a given arrangement of the highs, neither the low-4 nor the low-5 can be next to the high-3, so there are two positions and 2! arrangements. The other two lows may be in any of the two remaining positions and still produce a valid arrangement (2! arrangements). Therefore, there are

$$\frac{4! \times 2! \times 2!}{4} = 24$$

possible arrangements.

Case 4. Low: 1, 2, 3, 6 High: 4, 5, 7, 8

For a given arrangement of the lows, neither the high-4 nor the high-5 can be next to the low-6, so there are two positions and 2! arrangements. The other two highs may be in any of the two remaining positions and still produce a valid arrangement (2! arrangements). Therefore, there are

$$\frac{4! \times 2! \times 2!}{4} = 24$$

possible arrangements.

Case 5. Low: 1, 2, 4, 6 High: 3, 5, 7, 8

For a given position of the 6 (4 possible locations), the 3 must be in a non-adjacent location (2 possibilities). The 7 and 8 both must be next to the low-6, and the 1 and 2 both must be next to the high-3, but the 7 and 8 can switch with each other, and the 1 and 2 can also switch (2×2 possibilities). That leaves the low-4 and high-5, each with only one possible low or high location. Therefore, there are

$$\frac{4 \times 2 \times 2 \times 2}{4} = 8$$

possible arrangements.

Case 6. Low: 1, 2, 5, 6   High: 3, 4, 7, 8

The 7 and 8 are the only highs greater than the low 5 and 6, but the 7 and 8 cannot surround both the 5 and 6, so no arrangement is possible.

Therefore, there are a total of $144 + 72 + 24 + 24 + 8 = 272$ valid arrangements.

Solution 2 by: Gaku Liu (11/FL)

(a) Assume such a numbering exists. Let the numbers in the circles, starting from 1 going counterclockwise, be 1, a_1, a_2, $\ldots$, a_6. Since $a_1 > 1$, we must also have $a_1 > a_2$. It follows that $a_2 < a_3$, $a_3 > a_4$, and so on. In general, if i is odd, a_i is greater than its neighbors, and if i is even, a_i is less than its neighbors. Then $a_6 < 1$, a contradiction. So there are no possible numberings. This generalizes to any even number of circles.

(b) Let A_n be the answer to the general problem for n circles. We will find a recursive formula for odd n. Let the numbers in the circles, starting from 1 going counterclockwise, be 1, a_1, a_2, $\ldots$, a_{2m+1}, where m is a nonnegative integer. As in part (a), a_i is less than its neighbors if and only if i is even. Since there are no two numbers the number 2 can be greater than, $a_i = 2$ only if i is even.

Suppose $a_{2k} = 2$. Then the numbers 1 and 2 divide the larger circle into two arcs with $2k-1$ and $2m-2k+1$ circles each. The numbers 3 through 7 can be distributed between the two arcs in $\binom{2m}{2k-1}$ ways. Consider the numbers in the arc with $2k-1$ circles. Since every number is greater than 1 and 2, they can be arranged in the arc in the same number of ways as they can be arranged in the original diagram with $2k-1$ circles, which is A_{2k-1}. Similarly, the numbers in the other arc can be arranged in $A_{2m-2k+1}$ ways. Therefore, the total number of ways the empty circles can be filled in given $a_{2k} = 2$ is $\binom{2m}{2k-1}A_{2k-1}A_{2m-2k+1}$. Summing up the values for $k = 1, 2, \ldots, m$, we have

$$A_{2m+1} = \sum_{k=1}^{m} \binom{2m}{2k-1} A_{2k-1}A_{2m-2k+1}.$$

We have $A_1 = 1$, so $A_3 = \binom{2}{1}A_1^2 = 2$, $A_5 = \binom{4}{1}A_1A_3 + \binom{4}{3}A_3A_1 = 16$, and

$$A_7 = \binom{6}{1}A_1A_5 + \binom{6}{3}A_3^2 + \binom{6}{5}A_5A_1 = 6 \cdot 16 + 20 \cdot 2^2 + 6 \cdot 16 = 272.$$

Hence, the answer is 272.

Additional Comments. Using exponential generating functions, James Sundstrom derived that the the number of arrangements for $2n$ numbers is

$$\frac{2^{2n-1}(2^{2n}-1)|B_{2n}|}{n},$$

where B_n denotes the n^{th} Bernoulli number.