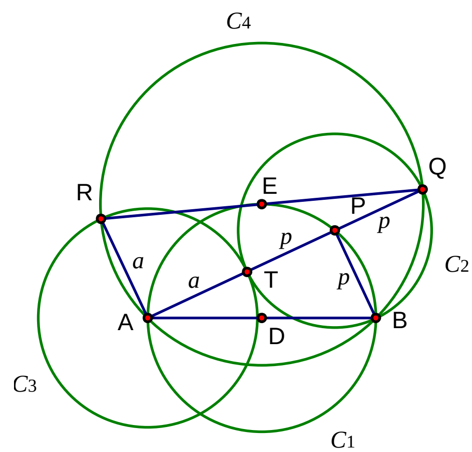
1/4/17. $\overline{AB}$ is a diameter of circle $\mathcal{C}_1$. Point P is on $\mathcal{C}_1$ such that $AP > BP$. Circle $\mathcal{C}_2$ is centered at P with radius PB. The extension of $\overline{AP}$ past P meets $\mathcal{C}_2$ at Q. Circle $\mathcal{C}_3$ is centered at A and is externally tangent to $\mathcal{C}_2$. R is on $\mathcal{C}_3$ such that $\overline{AR} \perp \overline{AQ}$. Circle $\mathcal{C}_4$ passes through A, Q, and R. Find, with proof, the ratio between the area of $\mathcal{C}_4$ and the area of $\mathcal{C}_1$, and show that this ratio is the same for all points P on $\mathcal{C}_1$ such that $AP > BP$.

Credit This problem was proposed by Ismor Fischer, University of Wisconsin.

Comments This problem can be easily solved by labeling lengths, and using right angles to apply Pythagoras's theorem. *Solutions edited by Naoki Sato.*

Solution 1 by: Matt Superdock (9/PA)

Let D be the center of C_1, and let E be the center of C_4. Let T be the point of tangency of C_2 and C_3. Let a be the radius of C_3, and let p be the radius of C_2. Then we have $AT = AR = a$ and $PB = PQ = PT = p$.



Since $\overline{AR} \perp \overline{AQ}$, $\angle RAQ$ is a right angle. Since right angles inscribe the diameter of the circle, RQ is the diameter of C_4. Since $\angle APB$ inscribes $\overline{AB}$, the diameter of C_1, $\angle APB$ is also a right angle.

The ratio we are seeking is

$$\frac{[C_4]}{[C_1]} = \frac{\pi(ER)^2}{\pi(AD)^2} = \frac{(QR)^2}{(AB)^2} = \frac{(AR)^2 + (AQ)^2}{(AP)^2 + (BP)^2} = \frac{(AR)^2 + (AT + TP + PQ)^2}{(AT + TP)^2 + (BP)^2}$$
$$= \frac{a^2 + (a + p + p)^2}{(a + p)^2 + p^2} = \frac{2a^2 + 4ap + 4p^2}{a^2 + 2ap + 2p^2} = \boxed{2}.$$

Since we made no assumptions, the ratio is 2 for all points P on C_1 such that $AP > BP$.