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Solutions to Problem 1/4/16
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1/4/16. Determine with proof the number of positive integers n such that a convex regular polygon with n sides has interior angles whose measures, in degrees, are integers.

Credit We are grateful to Professor Gregory Galperin, one of the world's most powerful problem posers, for suggesting this problem for the USAMTS program.

Comments Most students took the straightforward approach illustrated by Jake Snell and Kim Scott below. A few looked at the exterior angles instead of the interior angles, as shown by Zachary Abel. *Solutions edited by Richard Rusczyk.*

Solution 1 by: Jake Snell (11/NJ)
We know that the sum of interior angles in any n-gon is $180^\circ \times (n-2)$. Therefore, since we are considering only regular polygons, each interior angle is congruent and now

$$m(\text{each interior angle}) = \frac{180^\circ(n-2)}{n} = \frac{180^\circ n - 360^\circ}{n} = 180^\circ - \frac{360^\circ}{n}$$

Each interior angle will have an integer measure in degrees only if $\frac{360}{n}$ is an integer. Thus, n must be a factor of 360. We can construct these factors since we know that $360 = 2^3 \cdot 3^2 \cdot 5$. We seek all nonnegative integers a, b and c such that $2^a \cdot 3^b \cdot 5^c$ divides $2^3 \cdot 3^2 \cdot 5$. Since 2, 3, and 5 are all primes, $2^a | 2^3$, $3^b | 3^2$, and $5^c | 5$. $2^a | 2^3$ implies $\frac{2^3}{2^a} = 2^{(3-a)}$ is an integer. Therefore, $a \in \{0, 1, 2, 3\}$. In a similar manner, we find that $b \in \{0, 1, 2\}$ and $c \in \{0, 1\}$. Since $2^a \cdot 3^b \cdot 5^c$ is the prime factorization of a nonnegative integer, and since no two nonnegative numbers have the same prime factorization, given a unique combination of a, b and c, $2^a \cdot 3^b \cdot 5^c$ is a unique nonnegative integer. The number of unique combinations of a, b and c is simply the product of the number of different values they could be, or $4 \cdot 3 \cdot 2 = 24$. However, we must subtract 2 from this result since $1 = 2^0 \times 3^0 \cdot 5^0$ and $2 = 2^1 \cdot 3^0 \cdot 5^0$ do not yield polygons. Our answer is $n = 24 - 2 = 22$. //

Solution 2 by: Kim Scott (10/MA)

Since any polygon with $n \geq 3$ sides can be divided into $n-2$ triangles (each with total angle measure 180°), the total degree measure of its interior angles is $180(n-2)^\circ$. In a regular polygon, every interior angle has the same degree measure, which must thus be

$$\frac{180(n-2)}{n} = 180 - \frac{360}{n}$$

This is an integer exactly when $\frac{360}{n}$ is an integer, which is true iff n is a factor of 360.

Since $360 = 2^3 \times 3^2 \times 5$, its factors are of the form $2^a \times 3^b \times 5^c$, with $0 \leq a \leq 3$, $0 \leq b \leq 2$, and $0 \leq c \leq 1$. Since there are 4 values for a, 3 for b, and 2 for c, 360 has $4 * 3 * 2 = 24$ integral factors, and $\frac{360}{n}$ is an integer for 24 values of n. However, this count includes $n = 1$ and $n = 2$, which do not correspond to valid values for the number of sides of a regular polygon.

Therefore, there are $24 - 2 = 22$ positive integers n such that a convex regular polygon with n sides has interior angles whose measures, in degrees, are integers.

Solution 3 by: Zachary Abel (11/TX)

The interior angle is an integer if and only if the exterior angle is an integer because these two angles add to 180°. Since the exterior angle measures $360^\circ/n$, the condition holds if and only if n is a divisor of 360. Since $360 = 2^3 \cdot 3^2 \cdot 5$, this number has $4 \cdot 3 \cdot 2 = 24$ factors. But n must be at least 3, so we reject the possibilities that $n = 1$ or $n = 2$ and conclude that n may equal any of the other **22** divisors of 360.